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ALAN JEFFREY

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MAGNETOHYDRODYNAMICS

ALAN JEFFREY

Ph.D.

Professor of Engineering Mathematics
in the University of Newcastle upon Tyne

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PREFACE

In this book I have attempted to present the basic principles of magnetohydrodynamics which may be derived simply, and with reasonable mathematical rigour, from an elementary continuum model of a fluid. The reader is assumed to be familiar with the ideas leading up to the formulation of the Maxwell equations of electromagnetic theory and with vector methods. Although an understanding of fluid dynamics would be helpful, it is not strictly essential since any concepts needed are derived directly from first principles.

The picture of magnetohydrodynamics that can be presented by confining attention to a simple continuum approximation is, quite naturally, restricted in its applicability. Nevertheless, it still merits examination because it offers the easiest model by which the striking influence of magnetic fields on moving electrically-conducting fluids may be understood.

Many situations involving electrically-conducting gas flows require a more satisfactory description in terms of the Boltzmann equation used to describe plasma physics, but unfortunately such a discussion is beyond the scope of this book. However, it should be understood that the simpler explanation offered here suffices for the description of many geophysical and astrophysical phenomena and can also provide valuable approximations in various other situations.

The decision to discuss only those topics for which a straightforward and satisfactory mathematical argument exists has, of necessity, confined the treatment to certain aspects of magnetohydrodynamics. It is hoped, however, that the coverage is still fairly representative and that the simple physical discussions occasionally offered in place of a proper analysis will help to focus attention on matters requiring further examination should a more penetrating study be undertaken. Thus, for example, only a passing reference has been made to the stability of magnetohydrodynamic flows and shock waves. Also, only a simple physical description has been offered to indicate something of the many possible instability mechanisms that can occur to interfere with attempts at plasma containment by magnetic fields.

In order to supplement the text, and to show connections with electromagnetic theory and fluid dynamics, a few examples have been included which are not of direct relevance to magnetohydrodynamics. These examples have been indicated by an asterisk and may be disregarded at a first reading. The remaining examples have been chosen either to involve analysis that has only been briefly indicated in the text, or to provide a direct extension of the discussion in the text. For this reason these examples, which usually contain both a hint for the method of solution and the answer itself, should be attempted.

Following the accepted practice in the University Mathematical Texts, I have, with one exception, confined suitable cross-references to other publications in the same series. Because of this no proper form of acknowledgement to workers in the field of magnetohydrodynamics has been possible throughout the book. I am therefore taking advantage of the preface to acknowledge my debt of gratitude to Professors H. Grad, K. O. Friedrichs and J. Bazer of New York University, much of whose early work is described in this book. Their influence on my approach to magnetohydrodynamics, which took place largely during my stay at the Courant Institute of Mathematical Sciences in 1960-61, will be apparent to anyone familiar with their contributions.

Finally, I would like to express my gratitude to Professor D. E. Rutherford, Editor of this Series, for his early encouragement in the preparation of this manuscript and for his many suggestions for its improvement.

June 1965 A. J.

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CHAPTER I

THE FUNDAMENTAL EQUATIONS OF MAGNETOHYDRODYNAMICS

§ **1. Introduction.** Magnetohydrodynamics is the study of the macroscopic interaction of electrically-conducting liquids and gases with a magnetic field. As would be expected, both the equations of fluid mechanics and of electromagnetism feature in the description of magnetohydrodynamic flow; each impressing something of its own distinctive features on the subject. In our study of magnetohydrodynamics we shall assume familiarity only with vector analysis and with elementary electricity and magnetism extending as far as the formulation of the Maxwell equations. The necessary properties of fluid dynamics will be developed directly from the basic conservation laws of mass, momentum and energy. Reference will on occasion be made to pure hydrodynamical flows as limiting examples of magnetohydrodynamic flows, and at such times some familiarity with classical fluid dynamics would help to deepen understanding; this knowledge is, however, not essential. Hereafter the word fluid will be used to denote a continuum medium which may be either a liquid or a gas.

Although some of the features of magnetohydrodynamics are apparent when any ordinary electrically-conducting fluid such as mercury or liquid sodium moves in a magnetic field they are, in general, significant only in extremely high-temperature gases. All matter becomes ionised at sufficiently high temperatures forming a gas composed of individual ions and electrons. For ordinary gases ionisation occurs at temperatures in excess of 10^4 °K (Kelvin) while in the temperature range 10^6 °K to 10^9 °K, which typifies astrophysical phenomena, all matter is to be found in this state.

In the absence of a magnetic field a highly ionised gas behaves in most respects like a classical gas, but this behaviour is modified in a striking manner when a magnetic field is applied. Wave motions and fluid flows of a kind previously unknown occur and it is, for example, possible to generate waves in an ionised gas having features more in common with electromagnetic waves than with hydrodynamical waves, and so differing essentially from any disturbances that can occur in ordinary electrically non-conducting fluids. These remarkable effects have been used to account for many important physical phenomena, and considerable success has been attained in astrophysics where magnetohydrodynamic mechanisms have been suggested for the generation of the solar and terrestrial magnetic fields.

Although when matter is highly ionised ordinary chemical and physical properties are lost, it is, nevertheless, still in a highly complex state and its description must be phrased in terms of the energy exchanges and motions of the interacting particles. To describe these effects correctly would necessitate the detailed consideration of all the individual particles involved in the system: a prohibitively difficult task, involving as it does the microscopic behaviour of a very large number of interacting particles. The magnitude of the problem can be seen from the fact that the number density of particles in the outer layers of the sun's corona is approximately 10^8 particles/cm^3, while at room temperature and pressure air has approximately 3×10^{19} molecules/cm^3. To overcome this difficulty of description two quite different points of view may be adopted according to the nature of the problem; namely, the approaches of plasma physics and of magnetohydrodynamics. In plasma physics the observable properties of an ionised gas are described in terms of the collective behaviour of the individual particles forming the gas, while in magnetohydrodynamics a large-scale description of the behaviour of an electrically-conducting fluid is given in terms of a continuum approximation.

An ionised gas is called a **plasma** when the **Debye shielding length** λ_D in the ionised gas, that is the distance at which the electric field of a charge is shielded by neighbouring charges of opposite sign, is small compared with a representative length of interest. This important dimension provides an estimate of the maximum distance over which the local concentrations of positive and negative charge may differ appreciably from one another thereby causing a local departure from electrical neutrality. In a plasma in which ions and electrons are in thermodynamic equilibrium at an absolute temperature T having a number density N of electrons/cm^3, the Debye shielding length λ_D varies approximately as $(T/N)^{\frac{1}{2}}$ and, for a plasma at a temperature of 10^5 °K with a particle density of 10^{18}/cm^3, has the value $1{\cdot}6 \times 10^{-6}$ cm.

Some indication of the intensity of the electric field that acts to enforce the equality of electron and ion densities throughout the plasma may be easily obtained by considering the following simple example. We consider a sphere of radius $r = 1$ cm and assume that within it the aggregate electron charge exceeds the aggregate ion charge by 1 per cent., resulting in a total negative charge Q. Then, for an assumed electron density of 10^{14} electrons/cm^3, we have from Gauss's law that the electric field close to the surface of the sphere is

$$E = \frac{Q}{r^2} = \frac{4\pi}{3} \times 10^{14} \times 10^{-2} e,$$

where $e = 4{\cdot}8 \times 10^{-10}$ e.s.u., is the charge on an electron. So, near to the surface of the sphere, $E = 2 \times 10^3$ in electrostatic units of potential. To convert this result to the practical unit, the volt, we must now multiply by the conversion factor $c/10^8$, where $c = 3 \times 10^{10}$ cm/sec is the velocity of light in a vacuum. We then obtain $E = 6 \times 10^5$ volts.

The size of the electric field that is produced by this apparently small departure from electrical neutrality in the plasma shows that since the plasma is electrically conducting we must always assume that there is equality of electron and ion charge density when using scales of measurement in which the Debye shielding length is smaller than a representative length of interest. Thus is will always be assumed that no electric fields due to charge concentrations can exist within a plasma.

For our purposes we shall consider plasma physics to be the theory of ionised gases satisfying our definition in which the collective properties of the ionised particles are described in statistical terms, taking into account the properties of individual particles and the complicated energy exchange processes that can occur between them.

A satisfactory description of gases based on the statistical behaviour of individual gas particles was first produced by Boltzmann in 1872 in connection with studies of ordinary gases. This must be extended to include ionised gases if an adequate description of plasmas is to be obtained when they occur in complicated non-equilibrium environments of the type found in many experiments and in certain branches of astrophysics. A detailed examination of the Boltzmann equation for plasmas entails very considerable difficulties but is essential if a proper understanding of general plasma behaviour is to be achieved. Fortunately rather easier considerations provide useful criteria by which the applicability of simpler representations of electrically-conducting fluids may be judged. As may be expected, it sometimes happens that the statistical description of a plasma is sufficiently simple that by suitable averaging processes it may be replaced by a continuum representation. One criterion by which such an approximation may be allowed is when the electron mean free path for collisions l in the plasma is smaller than the **Larmor radius** r_e of the electrons in the plasma; that is the radius of the helical path followed by a free electron in the applied magnetic field $\boldsymbol{H}$.

This condition follows from the fact that when the electron Larmor radius is less than the electron mean free path for collisions, the magnetic effects become predominant in the equation describing electron behaviour and cause anistropic electrical conductivity in the plasma. The anisotropy that then occurs is due to the fact that if, under these conditions, an electric field acts such that it has a component perpendicular to the magnetic field $\boldsymbol{H}$ then, as well as the helical particle motions produced by the magnetic field, there is also an appreciable general drift of particles in a direction which is perpendicular to the plane determined by the magnetic field and the transverse electric field component. This then produces the surprising physical effect that the current that is caused to flow is not as might be anticipated from Ohm's law parallel to the applied electric field. The current produced by this anistropy in the electrical conductivity of the plasma is called the **Hall current.** This current can be detected in ordinary metallic conductors which are subjected to strong magnetic fields and it is named after E. H. Hall who, in 1880, was the first to detect such a current in a thin gold leaf subjected to mutually perpendicular magnetic and electric fields.

However, provided $r_e > l$, the Hall current is negligible and the electrical conductivity of the plasma may be considered to be a scalar. The current that then flows as a result of an electric field being applied to the plasma is parallel to the electric field; the relationship between current and electric field being approximated by Ohm's law. The electrical conductivity of plasmas can become extremely high and a good approximation to many problems may be achieved by assuming it to be infinite.

The electron Larmor radius in a plasma which is in thermodynamic equilibrium at an absolute temperature T, and is subjected to a magnetic field of strength H gauss, varies approximately as $(T/H^2)^{\frac{1}{2}}$. The value of r_e corresponding to a temperature of 10^5 °K and a magnetic field of strength $2{\cdot}5 \times 10^5$ gauss would be approximately $6{\cdot}1 \times 10^{-4}$ cm. The electron mean free path for collisions l is difficult to estimate but for our purposes may be assumed to vary approximately as $(T^2/N)[A+\log_e T\lambda_D]^{-1}$, where again N is the electron number density/cm^3 and A is a constant. If in the previous example $N = 10^{18}$, the electron mean free path for collisions would be $l = 4{\cdot}9 \times 10^{-4}$ cm. It will be readily seen that for low densities and high temperatures the electron mean free path for collisions can easily attain a value which is comparable to the dimensions of experimental apparatus. To prevent interaction with the walls of the containing vessel under these conditions it is necessary to contain the plasma by means of specially curved magnetic fields which attempt to confine particles to a closed volume within the walls of the apparatus.

When a plasma is considered to behave as a perfect gas with adiabatic exponent γ, the speed of sound a in the plasma, determined by the methods of the simple kinetic theory of gases, is $a = (\gamma kT/m)^{\frac{1}{2}}$ cm/sec, where $k = 1{\cdot}37 \times 10^{-16}$ erg/degree is the Boltzmann constant and m is the average particle mass. In the case of a plasma in thermal equilibrium comprising equal numbers of electrons and protons at a temperature 10^5 °K and having an adiabatic exponent $\gamma = 5/3$, the sound speed would be approximately $a = 5{\cdot}2 \times 10^6$ cm/sec.

A thorough study of the Boltzmann equation must be made in order to assess the relevance of all the important parameters occurring in plasma physics and of the conditions under which many different approximations are valid. However, the previous simple discussion based on average properties of a plasma must suffice to indicate for us the approximate behaviour of the three very important parameters λ_D, r_e and l, and to provide simple criteria by which we may judge the applicability of the magnetohydrodynamic approximation. So, when $r_e > l$ and, furthermore, the energy exchange processes in the plasma are in thermodynamic equilibrium, we have indicated that for dimensions in excess of λ_D the electrical and fluid properties of the plasma are of a conventional type appropriate to a continuum conducting fluid.

Accordingly then we assume these conditions to be satisfied, and take as the starting point of the magnetohydrodynamic description of a plasma the macroscopic continuum equations of fluid mechanics and of electromagnetic theory applied to an electrically conducting fluid.

Even within the simplified structure of magnetohydrodynamics further approximation is still possible and will need to be made from time to time in order to simplify a problem or to approximate a physical situation. Thus, when discussing the flow of an electrically-conducting fluid through a parallel channel with a magnetic field superimposed perpendicular to the channel walls, we shall find that we are able to solve the problem with the assumptions that the fluid is both viscous and has finite electrical conductivity; on other occasions, however, it will be necessary to consider only inviscid and perfectly conducting fluids if analytical results are to be obtained. Idealisations of this type are common in physics and applied mathematics and are usually required in order that an exact analytical representation of some aspect of a subject be obtained. Naturally it is highly desirable that these restrictions be removed wherever possible but this is difficult, and to do so usually necessitates recourse to approximate methods of solution, very often based on an intuitive physical understanding of the problem. We offer no discussion of these approximate methods but would remark that the understanding that is required for their formulation is derived only by familiarity with the simpler situations which we shall examine later in considerable detail. Accordingly then, we suggest that the discussion of magnetohydrodynamics that is presented here be regarded as something in the nature of an " asymptotic structure " to the subject, often directly applicable but sometimes describing only a limiting solution; frequently just such a limiting solution as would be used to test the correctness of an approximate solution to a real problem. We shall discuss some real gas effects and later it will be seen that a measure of the importance of real gas effects such as the viscosity and the electrical conductivity of the fluid in any given physical situation is provided by certain non-dimensional parameters that will be introduced. At this stage we will only remark that the assumption of an inviscid perfectly conducting fluid suffices for a number of practical purposes.

§ 2. The pre-Maxwell equations in a conductor at rest. In an arbitrary inertial system of coordinates † the Maxwell electromagnetic field equations, when expressed in terms of Gaussian units, take the form

$$\text{curl}\,\boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t}, \tag{2.1}$$

$$\text{curl}\,\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \tag{2.2}$$

$$\text{div}\,\boldsymbol{B} = 0, \tag{2.3}$$

$$\text{div}\,\boldsymbol{D} = 4\pi q, \tag{2.4}$$

† A system of space coordinates in which a free particle, which is subjected to no forces, moves in a straight line without acceleration.

in which t is time, $\boldsymbol{D}$ and $\boldsymbol{E}$ are the electric displacement and the electric field vectors, respectively, $\boldsymbol{H}$ is the magnetic field vector, $\boldsymbol{B}$ is the magnetic induction vector, $\boldsymbol{j}$ is the current density vector measured in e.s.u., c is the velocity of light and q is the charge density. We showed earlier that no electric fields can exist in the fluid due to local concentrations of charge and so, since most problems of interest involve only an external magnetic field we shall, unless otherwise stated, assume that no external electric field is present. Combining equations (2.1) and (2.4) immediately gives the **charge conservation law**

$$\frac{\partial q}{\partial t} + \operatorname{div} \boldsymbol{j} = 0. \tag{2.5}$$

This result is seen to be a direct consequence of the **field equations** of electromagnetism which express only those fundamental properties which are true for all electromagnetic continua and do not depend for their validity on any physical assumptions that may be made about a particular medium. When expressed in mathematical form, the physical assumptions that are taken to characterise a medium are called the **constitutive equations.** In a uniform isotropic medium, which is at rest relative to the inertial system, the quantities $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{H}$ are related by the linear constitutive equations

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}, \tag{2.6}$$

$$\boldsymbol{B} = \mu \boldsymbol{H}, \tag{2.7}$$

where ε is the dielectric constant and μ is the magnetic permeability, both of which are assumed to be constant. When the medium is also a homogeneous electrical conductor, the current flowing is the **conduction current** and, assuming **Ohm's law** as the appropriate constitutive equation, the conduction current $\boldsymbol{j}$ is proportional to the electric field vector $\boldsymbol{E}$, giving

$$\boldsymbol{j} = \sigma \boldsymbol{E}, \tag{2.8}$$

where σ is the **electrical conductivity**. If a free charge exists we must add to the right-hand side of equation (2.8) the convection current resulting from the motion of the charge. However, we shall show in the following discussion that one of the consequences of an assumption that is basic to magnetohydrodynamics is that a charge q cannot persist in a conductor at rest.

To see the nature of this important assumption and to establish our assertion let us combine equations (2.4) and (2.5) and then apply dimensional analysis to the result

$$\operatorname{div}\left(4\pi \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t}\right) = 0.$$

First, notice that the operator div has dimensions L^{-1}, where L is a representative length of interest, and the operator $\frac{\partial}{\partial t}$ has the dimensions of reciprocal time, T^{-1}, where T is a representative time of interest which, although we are not discussing periodic phenomena, it is often useful to interpret as a frequency ω of some phenomenon of interest. These quantities are of necessity somewhat arbitrary in their specification and, as we will show later, it is only a non-dimensional combination of them that is significant. However, to interpret ω a little more precisely, we might consider that the sound velocity a in a plasma represents a velocity of interest when, if L is a typical dimension in the experiment, $\omega = T^{-1} = aL^{-1}$. For a plasma with sound speed 5×10^6 cm/sec and a typical dimension $L = 10^2$ cm, $\omega = 5 \times 10^4$ sec^{-1}. Using the idea of a representative frequency ω we can use the constitutive equations (2.6) and (2.8) in the above equation to show that the condition for

$$\frac{4\pi}{c} |\boldsymbol{j}| \gg \frac{1}{c} \left| \frac{\partial \boldsymbol{D}}{\partial t} \right|$$

is that

$$\frac{\varepsilon\omega}{4\pi\sigma} \ll 1. \tag{I}$$

This is one of the basic assumptions that is made in magnetohydrodynamics, and in subsequent work we shall always assume that ω satisfies this condition. Since in most metallic conductors σ is of the order 5×10^{17} in e.s.u., the condition (I) is valid even when ω is close to the frequencies occurring in optical phenomena.

It is a direct consequence of the assumption (I) of magnetohydrodynamics that the displacement current $\frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t}$ occurring in Maxwell equation (2.1) can be neglected, thereby reducing the equation to

$$\text{curl}\,\boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{j}. \tag{2.9}$$

The system of equations (2.2), (2.3), (2.4) and (2.9) will be called the **pre-Maxwell equations** in the sense that they describe electromagnetic phenomena as they were understood before Maxwell introduced the displacement current term.† If, now, we take the divergence of equation (2.9) it follows at once that

$$\text{div}\,\boldsymbol{j} = 0, \tag{2.10}$$

so that in systems described by the pre-Maxwell equations the current vector $\boldsymbol{j}$ is **solenoidal** and all currents must flow in closed circuits. Using Ohm's law (2.8) and equations (2.4), (2.6) and (2.10) we then see that

$$\text{div}\,\boldsymbol{D} = 0, \text{ or } q = 0.$$

† See Coulson, *Electricity*, 1951, p. 224. Notice that $\boldsymbol{j}$ in equation (2.9) above is measured in e.s.u. To convert to e.m.u., as used by Coulson, replace $\frac{1}{c}\boldsymbol{j}$ by $\boldsymbol{j}$.

This establishes our assertion that the pre-Maxwell equations lead to the result that a charge cannot exist in a conductor which is at rest in an inertial system.

By taking the curl of equations (2.2) and (2.9), and using equations (2.7) and (2.8) together with the divergence conditions div $\boldsymbol{H} = 0$ and div $\boldsymbol{E} = 0$, we obtain

$$\nabla^2 \boldsymbol{H} = \frac{4\pi\sigma\mu}{c^2} \frac{\partial \boldsymbol{H}}{\partial t} \tag{2.11}$$

and

$$\nabla^2 \boldsymbol{E} = \frac{4\pi\sigma\mu}{c^2} \frac{\partial \boldsymbol{E}}{\partial t}. \tag{2.12}$$

These equations are parabolic in type and describe a diffusion process analogous to the flow of heat in a solid thus showing that the electromagnetic disturbance described by the pre-Maxwell equations diffuses through a conductor. A simple dimensional analysis of equations (2.11) and (2.12) then shows that the order of magnitude of the decay time, the time in which the field decays to $1/e$ of its initial value, is given by the expression $4\pi\sigma\mu L^2/c^2$, where L is a representative length of variation of the magnetic field.

§ 3. The electromagnetic field in a moving rigid conductor. Let us now consider the electromagnetic field in a slowly moving rigid conductor in a laboratory. A general discussion of this problem would involve the ideas of the special theory of relativity and would be out of place in this account. Instead, let us be content to show how, with one additional assumption, the non-relativistic equations may be derived.

The assumption which we now make in addition to condition (I), that $\varepsilon\omega \ll 4\pi\sigma$, is that a representative velocity V of the conductor is small compared with the velocity of light c, and hence that

$$\left(\frac{V}{c}\right)^2 \ll 1. \tag{II}$$

In the subsequent discussion we shall refer to the laboratory system of coordinates which will be assumed to be fixed and will be regarded as an inertial system of coordinates. Such a system is of course not necessarily an inertial system since, for example, the laboratory system of coordinates fixed relative to the earth is a rotating coordinate system relative to the sun. The Maxwell equations (2.1) to (2.4) are valid in any inertial system, irrespective of whether or not the conductor in question is moving or fixed with reference to the system of coordinates.† However, this is not the case for the constitutive equations (2.6) to (2.8) since they are expressions of relationships in the particular coordinate system that moves with the conductor.

Indeed, one non-invariant property of the constitutive equations of electromagnetic theory is already familiar in the form of the expression relating the electric field $\boldsymbol{E}$, that is experienced by a stationary observer, to the electric field $\boldsymbol{E}'$ experienced by an observer moving with velocity $\boldsymbol{v}$ in a magnetic field with magnetic induction vector $\boldsymbol{B}$. This expression,

$$\boldsymbol{E}' = \boldsymbol{E} + \frac{1}{c}(\boldsymbol{v} \times \boldsymbol{B}),$$

describes the Lorentz force experienced by a moving unit charge,‡ and is seen to depend on the motion of the observer relative to the fixed reference frame in which $\boldsymbol{E}$ is measured. The force is seen to comprise the component due to the field $\boldsymbol{E}$ together with the electromagnetic body force $\frac{1}{c}(\boldsymbol{v} \times \boldsymbol{B})$ due to the rate at which lines of force are crossed. Motion along the magnetic lines of force will not produce an electromagnetic body force.

It is a postulate of the special theory of relativity that a

† See Rindler, *Special Relativity*, 1960: the discussion on relativistic electrodynamics establishes this important property.

‡ See Coulson, *Electricity*, 1951, p. 121.

measuring rod, moving with speed v relative to another frame of reference, varies in length relative to that frame as the **Lorentz contraction factor** $\gamma = (1 - v^2/c^2)^{\frac{1}{2}}$ (see Examples 2 and 3 of § 12). By condition (II) above, this factor γ will become unity in the magnetohydrodynamic flows which we shall consider. The connections between the primed and the un-primed quantities in relativistic motion are then considerably simplified and, for example, the true relativistic relationship that exists between $\boldsymbol{E}$ and $\boldsymbol{E}'$, namely

$$\boldsymbol{E}' = \gamma\boldsymbol{E} + \frac{\boldsymbol{v}}{v^2}(\boldsymbol{v}\,.\,\boldsymbol{E})(1-\gamma) + \frac{\gamma}{c}(\boldsymbol{v}\times\boldsymbol{B}),$$

reduces to the non-relativistic result just discussed. Thus condition (II) is really just the condition that ensures that relativistic effects can be ignored.

Let us now consider a conductor moving with a constant velocity $\boldsymbol{v}$ relative to an inertial frame of reference, and denote by a prime the values of all quantities in the co-ordinate system moving with the conductor. Quantities in the inertial frame of reference are un-primed. Then, from the constitutive equations (2.6), (2.7) and (2.8), we have

$$\boldsymbol{D}' = \varepsilon\boldsymbol{E}', \tag{2.6'}$$

$$\boldsymbol{B}' = \mu\boldsymbol{H}', \tag{2.7'}$$

and

$$\boldsymbol{j}' = \sigma\boldsymbol{E}'. \tag{2.8'}$$

The relationships between the quantities $\boldsymbol{D}'$, $\boldsymbol{E}'$, $\boldsymbol{B}'$, $\boldsymbol{H}'$, $\boldsymbol{j}'$ and q' measured in the moving frame and the quantities $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{H}$, $\boldsymbol{j}$ and q measured in the laboratory system which we shall use, corresponding to the assumption that $\gamma = 1$, are †

$$\boldsymbol{D}' = \boldsymbol{D} + \frac{1}{c}(\boldsymbol{v}\times\boldsymbol{H}), \tag{3.1a}$$

† See Rindler, *loc. cit.*, and Tolman, R. C., *Relativity, Thermodynamics and Cosmology*, Oxford, 1932, § 52.

$$\boldsymbol{E}' = \boldsymbol{E} + \frac{1}{c}(\boldsymbol{v} \times \boldsymbol{B}), \tag{3.1b}$$

$$\boldsymbol{H}' = \boldsymbol{H} + \frac{1}{c}(\boldsymbol{v} \times \boldsymbol{D}), \tag{3.2a}$$

$$\boldsymbol{B}' = \boldsymbol{B} + \frac{1}{c}(\boldsymbol{v} \times \boldsymbol{E}), \tag{3.2b}$$

$$\boldsymbol{j}' = \boldsymbol{j} - q\boldsymbol{v}, \tag{3.3}$$

$$q' = q - \frac{1}{c^2}(\boldsymbol{v} \,.\, \boldsymbol{j}). \tag{3.4}$$

Now, because the dimensions of curl are L^{-1} and the dimensions of $\frac{\partial}{\partial t}$ are T^{-1}, we have from equation (2.2) the relation

$$L^{-1} \, |\boldsymbol{E}| \sim \frac{1}{c} T^{-1} \, |\boldsymbol{B}|,$$

leading to

$$|\boldsymbol{E}| \sim \frac{V}{c} |\boldsymbol{B}|, \tag{2.2'}$$

and so, since $v \sim V$, the third terms in equations (3.2) are of order $\left(\frac{V}{c}\right)^2$ and may be neglected. The condition $q = 0$ derived in § 2 causes the third term in equation (3.3) to vanish and, since it is true for any inertial system, gives as the condition on q' in the moving system

$$q' = 0. \tag{3.5}$$

In non-relativistic magnetohydrodynamics the constitutive equations (3.1) to (3.4) thus reduce to the following important set of equations

$$\boldsymbol{D}' = \boldsymbol{D} + \frac{1}{c}(\boldsymbol{v} \times \boldsymbol{H}), \tag{3.6a}$$

$$E' = E + \frac{1}{c}(v \times B), \tag{3.6b}$$

$$H' = H, \tag{3.7a}$$

$$B' = B, \tag{3.7b}$$

$$j' = j, \tag{3.8}$$

and

$$q' = q - \frac{1}{c^2}(v \, . j) = 0. \tag{3.9}$$

Introducing equations (3.6*b*) and (3.8) into equation (2.8′) gives

$$j = \sigma \left\{ E + \frac{1}{c}(v \times B) \right\}, \tag{3.10}$$

whilst from equations (3.6*a*) and (3.7*a*, *b*) we see that equation (2.7′) is invariant and that equation (2.6′) is transformed into

$$D = \varepsilon E + \frac{1}{c}(\varepsilon\mu - 1)(v \times H). \tag{3.11}$$

The equation in the moving conductor,

$$\operatorname{div} E' = \frac{q'}{\varepsilon} = 0,$$

is however not invariant under this transformation and becomes

$$\operatorname{div} E + \frac{\mu}{c}\{(\operatorname{curl} v) \, . H - (\operatorname{curl} H) \, . v\} = 0.$$

By using equation (2.9) together with the fact that v is a constant vector, this reduces to

$$\operatorname{div} E = \frac{4\pi\mu}{c^2} v \, . j$$

or, by equation (3.9), to

$$\operatorname{div} \boldsymbol{E} = 4\pi\mu q. \tag{3.12}$$

These results emphasise that it is meaningless to speak of the electric field or the charge without first specifying the coordinate system in which they are to be measured.

§ 4. **A moving deformable conductor.** So far we have assumed that the conductor is rigid and moves with a constant velocity. When the conductor is accelerated or is a deformable body such as a liquid or a gas the velocity $\boldsymbol{v}$ will be a function of space and time; consequently a system of coordinates moving with any part of the conductor will not be an inertial system. To apply our results to a fluid we must now consider the consequences of having a deformable conductor. In a system of coordinates which is not an inertial system the Maxwell equations do not take the form displayed in equations (2.1) to (2.4). However, even in this case we may consider an inertial system at an arbitrary point in the conductor which momentarily moves with that element of the conductor. Then, in each inertial system so defined, we still have the relations (2.6′) to (2.8′) and the transformation laws (3.1) to (3.4), but the assumptions (I) and (II) now no longer lead necessarily to the equation $q' = 0$. This result may easily be understood as follows. The expression $q' = 0$ was a consequence of the differential equations (2.4), (2.8) and (2.10) which assume the existence of an inertial system rigidly fixed in the conductor. Since no one of the inertial systems just defined can move with the conductor for all time, it follows that if a coordinate system moves with any point of the conductor for all time then it cannot in general be an inertial system.

When q' cannot be set equal to zero in equation (3.9) we cannot immediately neglect the convection current described by equation (3.9). Since, apart from this point, the situation is identical with that of the previous case, we still have the transformation laws (3.6*a*, *b*) and (3.7*a*, *b*) together with equation (3.9), where q' is now no longer equal to zero, while from equation (3.3)

$$\boldsymbol{j}' = \boldsymbol{j} - q\boldsymbol{v}. \tag{4.1}$$

Equation (2.8′) thus becomes

$$\boldsymbol{j} = q\boldsymbol{v} + \sigma\left\{\boldsymbol{E} + \frac{1}{c}(\boldsymbol{v}\times\boldsymbol{B})\right\}. \tag{4.2}$$

The consequences of these results may be seen as follows. First, from equations (2.4) and (3.11), we find that

$$\varepsilon \operatorname{div}\left\{\boldsymbol{E} + \frac{1}{c}(\boldsymbol{v}\times\boldsymbol{B})\right\} - \frac{1}{c}\operatorname{div}(\boldsymbol{v}\times\boldsymbol{H}) = 4\pi q.$$

Multiplying by $\omega/4\pi\sigma$ and using condition (I), we see that to within the accuracy implied by conditions (I) and (II),

$$q = -\frac{1}{4\pi c}\operatorname{div}(\boldsymbol{v}\times\boldsymbol{H}). \tag{4.3}$$

Using this value of q in equation (4.2) then gives

$$\boldsymbol{j} = \frac{-\boldsymbol{v}}{4\pi c}\operatorname{div}(\boldsymbol{v}\times\boldsymbol{H}) + \sigma\left\{\boldsymbol{E} + \frac{1}{c}(\boldsymbol{v}\times\boldsymbol{B})\right\}.$$

Thus, multiplying by $\frac{v}{c}$, noticing that $v \sim V$ and using (II), this reduces to our earlier equation

$$\boldsymbol{j} = \sigma\left\{\boldsymbol{E} + \frac{1}{c}(\boldsymbol{v}\times\boldsymbol{B})\right\}. \tag{3.10}$$

Hence, making the assumptions (I) and (II), we see that the current $\boldsymbol{j}$ is still given by expression (3.10) but that the charge q is related to the magnetic field $\boldsymbol{H}$ and the velocity $\boldsymbol{v}$ by equation (4.3). To see this relationship more clearly in terms of q' let us expand the right-hand side of equation (4.3) and use equation (2.9) to obtain

$$-\frac{1}{c}\boldsymbol{H}.\text{curl}\,\boldsymbol{v} = 4\pi\left(q - \frac{1}{c^2}\boldsymbol{v}.\boldsymbol{j}\right)$$

which, by equation (3.9), is equivalent to

$$q' = -\frac{1}{4\pi c}\boldsymbol{H}.\text{curl}\,\boldsymbol{v}. \qquad (4.3')$$

This last equation gives the charge q' in terms of the velocity $\boldsymbol{v}$ and the magnetic field strength $\boldsymbol{H}$. If the motion of the conductor is irrotational (curl $\boldsymbol{v} = \boldsymbol{0}$) we again have $q' = 0$.

§ 5. Energy of the electromagnetic field. Eliminating $\boldsymbol{E}$ between equations (2.2) and (3.10) and using equations (2.3), (2.7′), (2.9) and (3.7) we obtain the following equation in terms of $\boldsymbol{v}$ and $\boldsymbol{H}$,

$$\frac{c^2}{4\pi\mu\sigma}\nabla^2\boldsymbol{H} + \text{curl}\,(\boldsymbol{v}\times\boldsymbol{H}) - \frac{\partial\boldsymbol{H}}{\partial t} = \boldsymbol{0}. \qquad (5.1)$$

Equations (2.9), (3.10) and (4.3) will be regarded as the definitions of the current $\boldsymbol{j}$, the electric field $\boldsymbol{E}$ and the charge q, respectively. Equation (5.1) is the basic equation governing the electromagnetic field in an arbitrarily moving conductor. Multiplying this equation by the magnetic field vector $\boldsymbol{H}$ we find after some manipulation (see Examples 4 and 5, § 12) that it may be written in the form

$$\tfrac{1}{2}\frac{\partial \boldsymbol{H}^2}{\partial t} = \boldsymbol{v}.(\boldsymbol{H}\times\text{curl}\,\boldsymbol{H}) - \frac{4\pi}{\mu\sigma}\boldsymbol{j}^2 - \text{div}\left\{\boldsymbol{H}\times(\boldsymbol{v}\times\boldsymbol{H}) - \frac{c^2}{4\pi\mu\sigma}\boldsymbol{H}\times\text{curl}\,\boldsymbol{H}\right\}. \qquad (5.2)$$

Integrating this expression over a fixed volume V and using Gauss's divergence theorem and equation (2.9) we finally obtain

$$\frac{\partial}{\partial t}\int_V \frac{\mu \boldsymbol{H}^2}{8\pi}\,dV = \frac{\mu}{c}\int_V \boldsymbol{v}.(\boldsymbol{H}\times\boldsymbol{j})dV - \int_V \frac{\boldsymbol{j}^2}{\sigma}\,dV$$

$$-\frac{\mu}{4\pi}\int_S \left\{\boldsymbol{H}\times(\boldsymbol{v}\times\boldsymbol{H}) - \frac{c^2}{4\pi\mu\sigma}\boldsymbol{H}\times\text{curl}\,\boldsymbol{H}\right\}.d\boldsymbol{S}, \quad (5.3)$$

where S denotes the surface enclosing the volume V.

By virtue of relation (2.2′), the electric field energy density $\dfrac{\varepsilon \boldsymbol{E}^2}{8\pi}$ is negligible in comparison with the magnetic field energy density $\dfrac{\mu \boldsymbol{H}^2}{8\pi}$, and so equation (5.3) is the energy equation for the electromagnetic field.† The first term on the right-hand side of equation (5.3) is equal to the work done by the magnetic force $\dfrac{1}{c}\boldsymbol{j}\times\boldsymbol{B}$, the second term is the Joule loss which is observed as heat and the third term represents the flow of electromagnetic energy passing through the surface S. This energy flow will of course vanish if the magnetic field does not exist outside the volume under consideration. In such a case, it is also possible that the magnetic field energy increases with time if the first term exceeds the second. These two situations correspond, respectively, to a steady and to a growing magnetic field and are therefore of importance in connection with magnetohydrodynamic dynamo theories which attempt to explain the origin of stellar and terrestrial magnetic fields.

Equations (5.2) and (5.3) are sometimes expressed

† Compare with Coulson, *Electricity*, 1951, p. 230.

differently in terms of the Poynting vector †

$$\boldsymbol{P} = \frac{c}{4\pi}\, \boldsymbol{E} \times \boldsymbol{H}. \tag{5.4}$$

In the absence of dissipative effects this vector represents the energy flow per unit time across a fixed surface of unit area normal to $\boldsymbol{P}$, as may be seen directly from equation (5.2) when infinite conductivity is assumed, since it can then be written in the form

$$\operatorname{div} \boldsymbol{P} = -\frac{\partial}{\partial t}\left(\frac{\mu H^2}{8\pi}\right) - \frac{1}{c}\, \boldsymbol{v}.(\boldsymbol{j} \times \boldsymbol{B}). \tag{5.5}$$

The first term on the right represents the rate of decrease of the magnetic energy, while the second term represents the rate of doing work against the magnetic force $\frac{1}{c}\boldsymbol{j} \times \boldsymbol{B}$.

The integral form of this relation follows from equation (5.3) which, in the case of infinite conductivity, becomes

$$\int_S \boldsymbol{P}.d\boldsymbol{S} = -\frac{\partial}{\partial t}\int_V \frac{\mu H^2}{8\pi}\, dV - \frac{1}{c}\int_V \boldsymbol{v}.(\boldsymbol{j} \times \boldsymbol{B})dV. \tag{5.6}$$

When the volume V is moving this result must be slightly modified to include the change of magnetic energy due to the change of the volume V itself. The arguments involved are essentially similar to those that will be used in Chapter VI for the derivation of a rate of change theorem for vectors from which the shock conditions will be obtained. For a volume $V(t)$ moving with a velocity $\boldsymbol{u}$, not necessarily equal to the fluid velocity $\boldsymbol{v}$, the modification amounts to the addition of a term $\int_S \left(\frac{\mu H^2}{8\pi}\right) \boldsymbol{u}.d\boldsymbol{S}$ to the left-hand side of equation (5.6).

† Coulson, *loc. cit.*, p. 232.

§ 6. The basic equations of inviscid magnetohydrodynamics. Hereafter we shall assume that the conductor is a fluid or a plasma of the type discussed in the introduction. So far in our discussion we have taken the point of view that the motion of the conductor is given. However, it is obvious that the motion is affected by the electromagnetic field and due account must be taken of this. Let us assume for the moment that the flow velocity $\boldsymbol{v}(\boldsymbol{x}, t)$ of the conductor is given. The magnetic field determined by equation (5.1) gives rise to the current, the electric field and the charge through the equations (2.9), (3.10) and (4.3). On the other hand, under the electromagnetic field thus induced the charge and the current density induced at the same time lead to the electric force $q\boldsymbol{E}$ and the magnetic force $\frac{1}{c}\boldsymbol{j}\times\boldsymbol{B}$, respectively, which act on the substantial or moving element of the fluid. Thus it is clear that interaction may be expected between the fluid and the field.

The total force per unit volume $\boldsymbol{f}$ acting on the fluid is composed of three parts, the **electromagnetic force** $\boldsymbol{f}^{(em)}$, the **mechanical force** $\boldsymbol{f}^{(m)}$ and the **external force** $\boldsymbol{f}^{(ex)}$.

The electromagnetic force given by the equation

$$\boldsymbol{f}^{(em)} = q\boldsymbol{E} + \frac{1}{c}\boldsymbol{j}\times\boldsymbol{B}$$

reduces to

$$\boldsymbol{f}^{(em)} = \frac{1}{c}\boldsymbol{j}\times\boldsymbol{B}, \tag{6.1}$$

since, by dimensional arguments and the use of equation (2.2′), it is readily seen that the first term $q\boldsymbol{E}$ is of the order

$$\frac{\sigma}{c}\left(\frac{V}{c}\right)^3 |\boldsymbol{B}|^2$$

which, by (II), may be neglected in comparison with $\frac{1}{c}\boldsymbol{j}\times\boldsymbol{B}$.

In an inviscid fluid the mechanical force $\boldsymbol{F}$ acting on a fluid element with volume V and surface S is entirely determined by the action of the fluid pressure p acting on S. If the outward drawn vector surface element of area is $d\boldsymbol{S}$ then

$$\boldsymbol{F} = -\int_S p\,d\boldsymbol{S}$$

which, by the Gauss divergence theorem, becomes

$$\boldsymbol{F} = -\int_V \text{grad}\, p\,dV.$$

So, shrinking the volume V to the element dV, we see that the force exerted by the fluid on the element dV is $-\text{grad}\, p\,dV$. Hence the mechanical force $\boldsymbol{f}^{(m)}$ exerted by the inviscid fluid on a unit volume is

$$\boldsymbol{f}^{(m)} = -\text{grad}\, p. \tag{6.2}$$

The equation of motion of the fluid moving with velocity $\boldsymbol{v}$ can now be obtained by equating all the forces acting on the elementary volume dV of the fluid to the product of the mass $\rho\,dV$ of the fluid element and its acceleration $\frac{D\boldsymbol{v}}{Dt}$ giving, after cancellation of dV,

$$\rho\frac{D\boldsymbol{v}}{Dt} = \boldsymbol{f}^{(m)} + \boldsymbol{f}^{(em)} + \boldsymbol{f}^{(ex)}, \tag{6.3}$$

where $\frac{D}{Dt}$ denotes the substantial or total derivative † following the fluid motion

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\boldsymbol{v}\,.\,\text{grad}),$$

† See Rutherford, *Vector Methods*, 1954, p. 103. See also Rutherford, *Fluid Dynamics*, 1959, p. 6.

and ρ is the density of the fluid. Using equations (6.1) and (6.2) we finally obtain the **equation of fluid motion**

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\text{grad } p + \frac{\mu}{c} \boldsymbol{j} \times \boldsymbol{H} + \boldsymbol{f}^{(ex)}. \tag{6.4}$$

Let us now derive the equation expressing the conservation of mass of the fluid within an arbitrary volume V bounded by a surface S. To do this we equate the mass of fluid $\int_S \rho \boldsymbol{v} \, . \, d\boldsymbol{S}$ flowing out through vector surface element $d\boldsymbol{S}$ of the surface S to the loss of mass $-\frac{\partial}{\partial t}\int_V \rho dV$ from volume V, giving

$$\frac{\partial}{\partial t}\int_V \rho dV + \int_S \rho \boldsymbol{v} \, . \, d\boldsymbol{S} = 0.$$

Then applying the Gauss divergence theorem to the surface integral leads at once to the equation

$$\int_V \left\{ \frac{\partial \rho}{\partial t} + \text{div}\,(\rho \boldsymbol{v}) \right\} dV = 0$$

which, since V is an arbitrary volume, implies the following **continuity equation** expressing the conservation of mass of the fluid

$$\frac{\partial \rho}{\partial t} + \text{div}\,(\rho \boldsymbol{v}) = 0. \tag{6.5}$$

If, now, we add the term $\boldsymbol{v}\frac{\partial \rho}{\partial t}$ to both sides of equation (6.4) we can make use of the continuity equation (6.5) to express the equation of motion of the fluid in a different form. We find that

$$\frac{\partial(\rho \boldsymbol{v})}{\partial t} = -\text{grad } p + \frac{\mu}{c} \boldsymbol{j} \times \boldsymbol{H} - \rho(\boldsymbol{v}\, . \,\text{grad})\boldsymbol{v} - \boldsymbol{v}\,\text{div}\,(\rho \boldsymbol{v}) + \boldsymbol{f}^{(ex)}. \tag{6.6}$$

When the external force $f^{(ex)}$ that appears in equation (6.6) can be derived from a scalar potential † then the force field to which $f^{(ex)}$ belongs is **conservative** and equation (6.6) becomes an explicit expression of the **momentum conservation law.** This would, for example, be the case when $f^{(ex)}$ was an ordinary gravitational force. External forces will usually be neglected unless their effect is of special significance.

§ 7. The basic equations of viscous magnetohydrodynamics. We now consider how the equations describing inviscid flow require alteration to take into account the effect of viscosity. It is clear that the continuity equation (6.5) is valid for both viscid and inviscid fluids since it is simply a statement of the conservation of mass within an arbitrary volume and is independent of the nature of the fluid. However the momentum equation (6.6) requires modification since the physical effect of viscosity is to cause a transfer of momentum between adjacent elements of the fluid whenever their relative velocity is non-vanishing, and this effect has not so far been taken into account.

Whereas in studying the elastic behaviour of a solid it is the distortions of adjacent elements of the solid that are important, in the viscous flow of a fluid it is only the time rates of change of the distortions that are significant. This is apparent from the fact that when the fluid moves as a rigid body the relative velocity between particles is zero and the viscous forces must vanish; consequently they can only depend on the spatial derivatives of the components of the fluid velocity vector $\boldsymbol{v}$. The simplest possible assumption that we can make about the relationship that exists between the viscous forces and the space derivatives of $\boldsymbol{v}$ is that they are linear. We shall make this assumption since experiments have shown that this is a good approximation

† That is $f^{(ex)}$ is of the form $f^{(ex)} = -\text{grad}\ \phi$, where ϕ is a scalar function.

provided the relative velocities between adjacent fluid particles are small. In solids the internal force acting per unit plane area drawn in the material is called the **stress** acting on the area and depends on the orientation of the area. In fluids it is customary to speak both of the internal stresses in the fluid and of the negative stresses which are termed pressures. When a fluid is viscous the force acting on a unit plane area does not usually lie along the normal to the area. Rather than use the stress on such an area it proves to be more useful if it is resolved into components that lie along three chosen coordinate axes. The specification of these three components then completely describes the stress acting on the plane area in question. So, considering three mutually perpendicular unit plane areas at any point P in the fluid, we see that the force acting on an element of the fluid located at P is completely described by the nine components of stress associated with these three plane areas. The force per unit area on such a fluid element results from the effect of the fluid pressure p, acting normal to the area, together with the effect of the viscous forces. Adopting the Cartesian reference frame $O\{x_1, x_2, x_3\}$, we shall denote by $T_{ik}^{(m)}$ the x_k-component of the mechanical stress acting on a unit area drawn perpendicular to the x_i-axis and located at P, and decompose $T_{ik}^{(m)}$ as follows:

$$T_{ik}^{(m)} = -p\delta_{ik} + V_{ik}, \tag{7.1}$$

where V_{ik} is the viscous stress and δ_{ik} is the Kronecker delta.

We must now consider the form of V_{ik}. Since we have postulated that V_{ik} should depend linearly on the spatial derivatives of the components v_1, v_2 and v_3 of fluid velocity vector $\boldsymbol{v}$ it follows that in general V_{ik} must be a linear combination of all terms of the form $\dfrac{\partial v_p}{\partial x_q}$. As a detailed examination of the form of V_{ik} may be found elsewhere in texts on fluid dynamics† we shall now only use simple physical arguments to indicate how it is possible to arrive at the form of V_{ik}. First, since any change of volume of a fluid element will be important, it is apparent that a term proportional to div $\boldsymbol{v}$ must occur in each of the principal viscous stresses V_{ii}. This is in agreement with our assumption of a linear relationship in V_{ik} since

$$\operatorname{div} \boldsymbol{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}.$$

Secondly, since viscous forces must vanish for any rigid body motion of the fluid, and the most general such motion combines a translation and a rotation, it follows that the remaining terms of the form $\frac{\partial v_p}{\partial x_q}$ must vanish for all such rotations. This is equivalent to requiring that all combinations of such terms in V_{ik} must vanish when the velocity $\boldsymbol{v}$ at a point P is given by $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$, where $\boldsymbol{\omega}$ is the angular velocity of the body and $\boldsymbol{r}$ is the radius vector from the centre of rotation to P. Hence the only other terms that may occur in V_{ik} are of the form $(\partial v_i/\partial x_k + \partial v_k/\partial x_i)$. It is customary to combine these two types of terms occurring in V_{ik} in the following special way:

$$V_{ik} = \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \tfrac{2}{3}\delta_{ik} \operatorname{div} \boldsymbol{v}\right) + \zeta\delta_{ik} \operatorname{div} \boldsymbol{v}, \quad (7.2)$$

where η and ζ are the two coefficients of viscosity, both of which are positive and have dimensions M/LT. Here η is the coefficient of shear viscosity or, as it is sometimes called, the **coefficient of dynamic viscosity** and ζ is the **coefficient of bulk viscosity.** These coefficients are temperature dependent but for simplicity we shall assume that they remain constant. Since ζ is usually very much smaller than

† See Rutherford, *Fluid Dynamics*, 1959, § 56.

η it is frequently neglected in discussions of viscous flow. The (3 × 3) array formed by the quantities V_{ik} will be called the **viscosity stress tensor** while the (3 × 3) array $T_{ik}^{(m)}$, where $T_{ik}^{(m)}$ is given by

$$T_{ik}^{(m)} = -p\delta_{ik} + \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \tfrac{2}{3}\delta_{ik} \operatorname{div} \boldsymbol{v}\right) + \zeta\delta_{ik} \operatorname{div} \boldsymbol{v}, \quad (7.3)$$

will be called the **mechanical tensor.** When the fluid is inviscid $\eta = \zeta = 0$ and the mechanical tensor simply describes the pressure. The modification to equation (6.6) that is necessary for it to describe viscous flow is now clear: each of the three components of grad p occurring in equation (6.6) must be replaced by the sum of the gradients of the appropriate three components of the stress tensor $T_{ik}^{(m)}$. When this is done (Example 6, § 12) the **momentum equation for viscous flow** becomes

$$\frac{\partial(\rho \boldsymbol{v})}{\partial t} = -\operatorname{grad} p + \frac{\mu}{c}\boldsymbol{j} \times \boldsymbol{H} + \eta\nabla^2 \boldsymbol{v} + (\zeta + \tfrac{1}{3}\eta) \operatorname{grad} \operatorname{div} \boldsymbol{v}$$

$$-\boldsymbol{v} \operatorname{div}(\rho \boldsymbol{v}) - \rho(\boldsymbol{v} \cdot \operatorname{grad})\boldsymbol{v} + \boldsymbol{f}^{(ex)}. \quad (7.4)$$

Alternatively, the equation analogous to (6.4), the equation of motion, is

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\operatorname{grad} p + \frac{\mu}{c}\boldsymbol{j} \times \boldsymbol{H} + \eta\nabla^2 \boldsymbol{v} + (\zeta + \tfrac{1}{3}\eta) \operatorname{grad} \operatorname{div} \boldsymbol{v} + \boldsymbol{f}^{(ex)}, \quad (7.5)$$

which is a generalisation of the **Navier-Stokes equation** of classical fluid dynamics.†

It is also possible to display the electromagnetic force $\boldsymbol{f}^{(em)}$, which is the same for both viscous and inviscid flows, in the form of an electromagnetic stress tensor.

To do this we make use of equation (2.9) to re-write

† See Rutherford, *Fluid Dynamics*, 1959, p. 201.

equation (6.1) in the form

$$\boldsymbol{f}^{(em)} = \frac{\mu}{4\pi}(\text{curl}\,\boldsymbol{H})\times\boldsymbol{H} \tag{6.1'}$$

and use the identity grad $\boldsymbol{H}^2 = 2(\boldsymbol{H}.\text{grad})\boldsymbol{H}+2\boldsymbol{H}\times\text{curl}\,\boldsymbol{H}$ to obtain

$$\boldsymbol{f}^{(em)} = \frac{\mu}{4\pi}[(\boldsymbol{H}.\text{grad})\boldsymbol{H}-\tfrac{1}{2}\,\text{grad}\,\boldsymbol{H}^2]. \tag{7.6}$$

Then, defining

$$T_{ik}^{(em)} \equiv \frac{\mu}{4\pi}(H_iH_k-\tfrac{1}{2}H^2\delta_{ik}). \tag{7.7}$$

it follows that the x_i-component $f_i^{(em)}$ of $\boldsymbol{f}^{(em)}$ is given by

$$f_i^{(em)} = \sum_{k=1}^{3}\frac{\partial T_{ik}^{(em)}}{\partial x_k}. \tag{7.8}$$

The (3×3) array formed by the quantities $T_{ik}^{(em)}$ will be given the name the **Maxwell stress tensor.**

§ **8. Thermodynamical considerations.** Since a fluid in different thermodynamics states will behave in markedly different ways it follows that when supplementing the conservation equations of mass and momentum by the addition of an energy equation, due account must be taken of the thermodynamical environment of the fluid. The energy conservation law is equivalent to the thermodynamical law †

$$TdS = de+pd\tau, \tag{8.1}$$

where T is the temperature, e is the internal energy per unit mass of the fluid, S is the entropy per unit mass of the fluid and $\tau = 1/\rho$ is the specific volume of the fluid (i.e., the volume per unit mass of the fluid). In a **reversible process** TdS can be considered as the heat per unit mass that is

† See Rutherford, *Fluid Dynamics*, 1959, p. 147.

gained by the fluid due to conduction, while in an **irreversible process** other sources of heat such as viscosity are present which result in TdS exceeding the heat acquired by conduction. The state of a gas is defined by the quantities p, T, τ, S and e, only two of which are independent. It is usually most convenient to express all quantities in terms of τ and S, and for a gas in which the dependence of e on τ and S is known, it follows at once from equation (8.1) that $T = \partial e/\partial S$ and $p = -\partial e/\partial \tau$. In our work we shall consider an irreversible process to be one involving the effects of viscosity and of electrical resistance and we shall interpret the difference between TdS and the heat acquired by conduction as the heat resulting from the effects of viscous forces and of Joule loss. Equation (5.3) shows that the heat per unit volume per unit time resulting from Joule loss is equal to $\boldsymbol{j}^2/\sigma$, whereas in ordinary hydrodynamics † the heat due to viscous forces is equal to

$$\eta \sum_{i,k=1}^{3} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \tfrac{2}{3}\delta_{ik} \operatorname{div} \boldsymbol{v}\right) \frac{\partial v_i}{\partial x_k}$$

per unit volume, per unit time, while the heat acquired by conduction is

$$\operatorname{div} (\chi \operatorname{grad} T)$$

per unit volume, per unit time, where χ denotes the thermal conductivity. From equation (8.1) we see that the increase of heat per unit time of a unit mass of fluid as it moves in space is $T\dfrac{DS}{Dt}$, and so the heat increase per unit volume, $\rho T\dfrac{DS}{Dt}$, may be equated to the heat influx due to viscous dissipation, Joule heating and thermal conduction per unit

† See Rutherford, *loc. cit.*, § 57. Notice that both here and subsequently we assume that $\zeta = 0$; a condition which is valid for most viscous fluids which are adequately described by specifying η.

volume per unit time, to give

$$\rho T \frac{DS}{Dt} = \eta \sum_{i,k=1}^{3} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \tfrac{2}{3}\delta_{ik} \operatorname{div} \boldsymbol{v} \right) \frac{\partial v_i}{\partial x_k} + \frac{\boldsymbol{j}^2}{\sigma} + \operatorname{div}(\chi \operatorname{grad} T). \tag{8.2}$$

Making use of the fact that $\tau = 1/\rho$ enables the thermodynamical law (8.1) to be rewritten in the alternative form

$$\rho T \frac{DS}{Dt} = \rho \frac{De}{Dt} - \frac{p}{\rho} \frac{D\rho}{Dt}.$$

Combining this result with equation (8.2) and using the continuity equation (6.5) expressed in its alternative form

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \boldsymbol{v} = 0, \tag{8.3}$$

we find that

$$\rho \frac{De}{Dt} = -p \operatorname{div} \boldsymbol{v} + \eta \sum_{i,k=1}^{3} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \tfrac{2}{3}\delta_{ik} \operatorname{div} \boldsymbol{v} \right) \frac{\partial v_i}{\partial x_k} + \frac{\boldsymbol{j}^2}{\sigma} + \operatorname{div}(\chi \operatorname{grad} T). \tag{8.4}$$

Now the total energy U per unit volume is

$$U = \tfrac{1}{2}\rho \boldsymbol{v}^2 + \frac{\mu \boldsymbol{H}^2}{8\pi} + \rho e, \tag{8.5}$$

where the first term on the right-hand side is the contribution due to the kinetic energy of the fluid, the second term is the energy density of the magnetic field and the third term is the internal energy density of the fluid. Assuming that $\boldsymbol{f}^{(ex)} = \boldsymbol{0}$, differentiating equation (8.5) with respect to t and making use of equations (2.9), (5.2), (6.5), (7.5) and (8.4), we finally obtain the following expression of the **energy conservation law** (Example 7, § 12)

$$\frac{\partial U}{\partial t} + \operatorname{div} \boldsymbol{g} = 0 \tag{8.6}$$

in terms of the energy flow vector $\boldsymbol{g}$. The vector $\boldsymbol{g}$ is given by the equation

$$\boldsymbol{g} = \rho \boldsymbol{v}(\tfrac{1}{2}v^2 + i) + \frac{\mu}{4\pi}\left\{\boldsymbol{H}\times(\boldsymbol{v}\times\boldsymbol{H}) - \frac{c^2}{4\pi\mu\sigma}\boldsymbol{H}\times\operatorname{curl}\boldsymbol{H}\right\}$$

$$-\eta \sum_{i,k=1}^{3}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \tfrac{2}{3}\delta_{ik}\operatorname{div}\boldsymbol{v}\right) v_i \boldsymbol{e}_k - \chi \operatorname{grad} T, \tag{8.7}$$

where $i = e + p\tau$ is called the **enthalpy** per unit mass of the fluid and where $\boldsymbol{e}_i$ is the unit vector parallel to the x_i-axis.

The conservation laws (6.5), (8.6) and (7.4) or (7.5) together with equation (5.1) constitute the basic equations of magnetohydrodynamics. When these conservation laws are supplemented by the addition of the constitutive equations of state such as

$$p = p(\rho, T), \quad e = e(\rho, T)$$

describing the properties of a particular fluid, the resulting set of equations then completely determines the behaviour of the interacting fluid and magnetic field when proper boundary conditions and initial conditions are prescribed. The conditions in a fluid are said to be **adiabatic** if there is no energy transport to or from any fluid element and the region external to the fluid.

For most purposes it is possible to assume that the fluids under consideration behave like an **ideal gas**, and hence that their **equation of state** is expressed by the law

$$p\tau = RT, \tag{8.8}$$

where R is the universal gas constant. The internal energy e of an ideal gas is dependent only on the temperature T and when, as is usually the case, the energy is assumed to be proportional to the temperature T, the gas is called **polytropic.** The consequences of this assumption are very important and are sufficient to determine the dependence of p on ρ and S as we now show.

The **specific heat at constant volume,** c_v, of a gas is defined to be the limit of the ratio of the energy supplied to a unit mass of gas, $T\delta S$, to δT, in raising its temperature an amount δT while keeping the volume constant. From equation (8.1) we see that $c_v = \partial e/\partial T$. Clearly, for a polytropic gas, we may write $e = c_v T$. The **specific heat at constant pressure,** c_p, is similarly defined only here it is the gas pressure instead of the gas volume that is kept constant during the addition of energy. For a polytropic gas it follows from equation (8.1) that $c_p = c_v + p\partial\tau/\partial T$ or, since from equation (8.8) we may write $p\partial\tau/\partial T = R$, this may be written as

$$R = c_p - c_v. \tag{8.9}$$

By writing equation (8.1) in the form

$$TdS = c_v dT + pd\tau,$$

and combining it with the differential form of equation (8.8) we find that

$$pd\tau + \tau dp = RdT.$$

Then, using equation (8.9), we arrive at the equation

$$dS = c_v \frac{dp}{p} + c_p \frac{d\tau}{\tau}. \tag{8.10}$$

Setting $c_p/c_v = \gamma$, the **adiabatic exponent** of the gas, equation (8.10) may be integrated in terms of given initial conditions, which we denote by a suffix O, and written in the form

$$p = A(S)\rho^{\gamma}, \tag{8.11}$$

where

$$A(S) = p_0 \tau_0^{\gamma} \exp\{(S - S_0)/c_v\}. \tag{8.12}$$

This shows the important fact that for a polytropic gas the coefficient $A(S)$ in equation (8.11) is a function only of the entropy S and does not depend on the nature of the gas.

It is easy to see from the results so far obtained that the fluid and magnetic field interaction which we have discussed is completely different from the primary interaction between the magnetic field and the charged particles which constitute the conducting medium. Namely, the interaction considered so far is essentially due to the induction resulting from the motion of a conducting fluid across the magnetic field. The current and charge do not play a primary role as sources of the electromagnetic field, but are given as consequences of the electric and magnetic fields produced by the induction mechanism.

§ 9. Summary of basic magnetohydrodynamic equations. Let us now summarise the basic equations and assumptions of magnetohydrodynamics.

The equations of motion of the fluid

Mass conservation:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}) = 0. \tag{6.5}$$

Momentum conservation: $(\boldsymbol{f}^{(ex)} = \boldsymbol{0})$

$$\frac{\partial(\rho \boldsymbol{v})}{\partial t} = -\operatorname{grad} p + \frac{\mu}{c} \boldsymbol{j} \times \boldsymbol{H} + \eta \nabla^2 \boldsymbol{v} + (\zeta + \tfrac{1}{3}\eta) \operatorname{grad} \operatorname{div} \boldsymbol{v} - \boldsymbol{v} \operatorname{div}(\rho \boldsymbol{v}) - \rho(\boldsymbol{v} \cdot \operatorname{grad})\boldsymbol{v} \tag{7.4}$$

or

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\operatorname{grad} p + \frac{\mu}{c} \boldsymbol{j} \times \boldsymbol{H} + \eta \nabla^2 \boldsymbol{v} + (\zeta + \tfrac{1}{3}\eta) \operatorname{grad} \operatorname{div} \boldsymbol{v}. \tag{7.5}$$

Energy conservation:

$$\frac{\partial U}{\partial t} + \operatorname{div} \boldsymbol{g} = 0 \tag{8.6}$$

with

$$U = \tfrac{1}{2}\rho v^2 + \frac{\mu H^2}{8\pi} + \rho e \tag{8.5}$$

and

$$g = \rho v(\tfrac{1}{2}v^2 + i) + \frac{\mu}{4\pi}\left\{H\times(v\times H) - \frac{c^2}{4\pi\mu\sigma} H\times \operatorname{curl} H\right\}$$

$$-\eta \sum_{i,k=1}^{3}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \tfrac{2}{3}\delta_{ik}\operatorname{div} v\right) v_i e_k - \chi \operatorname{grad} T \tag{8.7}$$

or, alternatively,

$$\rho T\frac{DS}{Dt} = \eta \sum_{i,k=1}^{3}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \tfrac{2}{3}\delta_{ik}\operatorname{div} v\right)\frac{\partial v_i}{\partial x_k}$$

$$+\frac{j^2}{\sigma} + \operatorname{div}(\chi \operatorname{grad} T). \tag{8.2}$$

The equations for the electromagnetic field in the conductor

Equation for the magnetic field:

$$\frac{\partial H}{\partial t} = \operatorname{curl}(v\times H) + \frac{c^2}{4\pi\mu\sigma}\nabla^2 H \tag{5.1}$$

$$\operatorname{div} B = \operatorname{div} H = 0. \tag{2.3}$$

Equation for current:

$$j = \frac{c}{4\pi}\operatorname{curl} H. \tag{2.9}$$

Equation for electric field:

$$E = \frac{1}{\sigma}j - \frac{\mu}{c}v\times H. \tag{3.10}$$

Equation for charge:

$$q = \frac{-1}{4\pi c}\operatorname{div}(v\times H). \tag{4.3}$$

Constitutive equations:

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} + \frac{1}{c}(\varepsilon\mu - 1)(\boldsymbol{v} \times \boldsymbol{H}) \tag{3.11}$$

$$\boldsymbol{B} = \mu \boldsymbol{H}. \tag{2.7}$$

The basic assumptions of non-relativistic magnetohydrodynamics

$$\frac{\varepsilon\omega}{4\pi\sigma} \ll 1 \tag{I}$$

$$\left(\frac{V}{c}\right)^2 \ll 1. \tag{II}$$

The constitutive equations in the coordinate system momentarily at rest in the conductor

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}, \quad \boldsymbol{B} = \mu \boldsymbol{H}, \quad \boldsymbol{j} = \sigma \boldsymbol{E}.$$

In order to illustrate how the basic equations determine the flow and the field we now assume that there is no energy gain to the fluid from external sources and that no viscous forces act, and hence that the fluid is adiabatic and reversible. These assumptions then imply that $\eta = 0$, $\zeta = 0$, $\chi = 0$ and $\sigma = \infty$ and that p is given by the constitutive equation of a polytropic gas

$$p = A(S)\rho^{\gamma}, \tag{9.1}$$

where A is a function only of entropy and γ is the adiabatic exponent.

The previous results then simplify to the following set of equations which we shall call the **Lundquist equation:**

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}) = 0 \tag{6.5}$$

$$\rho \frac{D\boldsymbol{v}}{Dt} = \frac{\mu}{4\pi}(\operatorname{curl} \boldsymbol{H}) \times \boldsymbol{H} - \operatorname{grad} p(\rho, S) \tag{7.5'}$$

$$\frac{DS}{Dt} = 0 \tag{8.2$'$}$$

$$\frac{\partial \boldsymbol{H}}{\partial t} = \text{curl}\,(\boldsymbol{v} \times \boldsymbol{H}) \tag{5.1$'$}$$

$$\text{div}\, \boldsymbol{H} = 0, \tag{2.3}$$

the current $\boldsymbol{j}$ being given by equation (2.9).

The eight scalar equations described by the Lundquist equations are frequently used in magnetohydrodynamics and constitute a consistent system of equations for the two scalar quantities ρ and S and for the six components of $\boldsymbol{v}$ and $\boldsymbol{H}$. Equation (2.3) can be regarded as a constraint for the initial configuration of the magnetic field $\boldsymbol{H}$. To show this we need only take the divergence of equation (5.1′) to see that the condition (2.3) is automatically satisfied for all time if it is satisfied initially.

§ 10. Basic properties of the magnetic field. Let us now derive an important theorem concerning the rate of change of the magnetic flux through a surface moving with the fluid. To do this let us consider a closed curve Γ in space and assume that Γ moves with the velocity $\boldsymbol{v}(\boldsymbol{x}, t)$ of each fluid particle associated with Γ. Further, let S denote a smooth surface spanning Γ and let each point of S also move with the fluid particles associated with S. During the time increment δt the curve Γ_1 and the surface S_1 at time t move to become Γ_2 and S_2, respectively, at time $t+\delta t$.

Now the **magnetic flux** † ϕ through the surface S_1 is

$$\phi = \int_{S_1} \boldsymbol{H}.d\boldsymbol{S}_1, \tag{10.1}$$

where dS_1 is an element of the area S_1 and $d\boldsymbol{S}_1 = \boldsymbol{n}dS_1$ with $\boldsymbol{n}$, the normal to dS_1, chosen in the same sense as the

† See Coulson, *Electricity*, 1951, p. 119.

fluid velocity vector $\boldsymbol{v}$. The change $\delta\phi$ in the flux ϕ consequent on the surface S_1 moving in time δt to S_2 is

$$\delta\phi = \int_{S_2} \boldsymbol{H}(t+\delta t)\,.\,d\boldsymbol{S}_2 - \int_{S_1} \boldsymbol{H}(t)\,.\,d\boldsymbol{S}_1, \qquad (10.2)$$

or, since to the first order we may write

$$\boldsymbol{H}(t+\delta t) = \boldsymbol{H}(t) + \left(\frac{\partial \boldsymbol{H}}{\partial t}\right)\delta t,$$

$$\delta\phi = \int_{S_2} \boldsymbol{H}(t)\,.\,d\boldsymbol{S}_2 - \int_{S_1} \boldsymbol{H}(t)\,.\,d\boldsymbol{S}_1 + \delta t \int_{S_2} \left(\frac{\partial \boldsymbol{H}}{\partial t}\right).\,d\boldsymbol{S}_2. \qquad (10.3)$$

The infinitesimal displacement undergone by each point P of Γ_1 to become a point P' of Γ_2 is clearly $\boldsymbol{v}\delta t$, and we shall denote by S_3 the surface ruled by all such vectors $\overline{PP'}$ (see Fig. 1). Then, adding to and substracting from the right-hand side of equation (10.3) the flux through S_3, and reversing the sense of $d\boldsymbol{S}_1$ so that it points outwards from the volume V enclosed by the surface $S_1+S_2+S_3$ we obtain, to the first order,

$$\delta\phi = \int_{S_1+S_2+S_3} \boldsymbol{H}(t)\,.\,d\boldsymbol{S} - \int_{S_3} \boldsymbol{H}(t)\,.\,d\boldsymbol{S}_3 + \delta t \int_{S_2} \left(\frac{\partial \boldsymbol{H}}{\delta t}\right).\,d\boldsymbol{S}_2, \qquad (10.4)$$

where $d\boldsymbol{S}$ is now an element of the surface bounding V. However by the Gauss divergence theorem we may rewrite the first term of equation (10.4) as $\int_V \operatorname{div} \boldsymbol{H}\, dV$, which vanishes by virtue of equation (2.3). Choosing the sense of the element $d\boldsymbol{r}$ of curve Γ_1 so that $d\boldsymbol{S}_3 = (\boldsymbol{v}\times d\boldsymbol{r})\delta t$ we see that the second term of equation (10.4) becomes

$$-\int_{S_3} \boldsymbol{H}\,.\,d\boldsymbol{S}_3 = -\delta t \oint_{\Gamma_1} \boldsymbol{H}\,.\,(\boldsymbol{v}\times d\boldsymbol{r}).$$

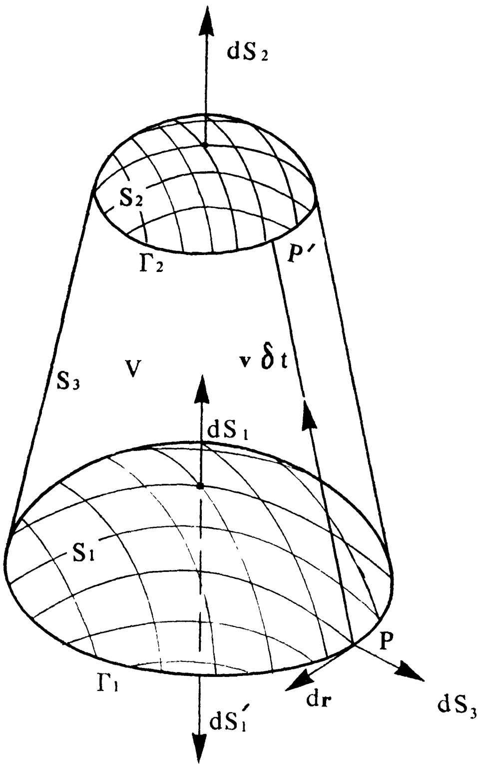

FIG. 1.

Since $\boldsymbol{H}.(\boldsymbol{v}\times d\boldsymbol{r}) = d\boldsymbol{r}.(\boldsymbol{H}\times\boldsymbol{v})$ we may apply Stokes's theorem † to this result to obtain

$$-\int_{S_3} \boldsymbol{H}.d\boldsymbol{S}_3 = -\delta t\int_{S_1} \operatorname{curl}(\boldsymbol{H}\times\boldsymbol{v}).d\boldsymbol{S}_1'$$

where, by the conditions of Stokes's theorem, $d\boldsymbol{S}_1'$ is an element of S_1 and is oriented so that a vector drawn from the interior of S_1 to Γ_1, an element $d\boldsymbol{r}$ of Γ_1 and $d\boldsymbol{S}_1'$, taken in that order, together form a right-handed set. Consequently $d\boldsymbol{S}_1' = -d\boldsymbol{S}_1$, and so

$$-\int_{S_3} \boldsymbol{H}.d\boldsymbol{S}_3 = \delta t\int_{S_1} \operatorname{curl}(\boldsymbol{H}\times\boldsymbol{v}).d\boldsymbol{S}_1.$$

Thus, equation (10.4) may be approximated to the first order by

$$\delta\phi = \delta t\int_{S_1}\left\{\frac{\partial\boldsymbol{H}}{\partial t}+\operatorname{curl}(\boldsymbol{H}\times\boldsymbol{v})\right\}.d\boldsymbol{S}_1.$$

So, dividing by δt and taking the limit as $\delta t\to 0$, we find that

$$\frac{D}{Dt}\int_{S_1}\boldsymbol{H}.d\boldsymbol{S}_1 = \int_{S_1}\left\{\frac{\partial\boldsymbol{H}}{\partial t}+\operatorname{curl}(\boldsymbol{H}\times\boldsymbol{v})\right\}.d\boldsymbol{S}_1. \quad (10.5)$$

We now substitute for $\partial\boldsymbol{H}/\partial t$ from equation (5.1) to obtain,

$$\frac{D}{Dt}\int_{S_1}\boldsymbol{H}.d\boldsymbol{S}_1 = \eta_m\int_{S_1}(\nabla^2\boldsymbol{H}).d\boldsymbol{S}_1, \quad (10.6)$$

where $\eta_m = c^2/4\pi\mu\sigma$ is the **magnetic viscosity.** Since $\operatorname{curl}(\operatorname{curl}\boldsymbol{H}) = \operatorname{grad}(\operatorname{div}\boldsymbol{H})-\nabla^2\boldsymbol{H}$ and since, by equation (2.3), $\operatorname{div}\boldsymbol{H} = 0$, we have

$$\operatorname{curl}(\operatorname{curl}\boldsymbol{H}) = -\nabla^2\boldsymbol{H}$$

† See Rutherford, *Vector Methods*, 1954, p. 74. Notice that although some authors use the symbol $\oint$ to denote both a line integral and a surface integral we shall reserve it exclusively for a line integral.

or, using equation (2.9),

$$\frac{4\pi}{c}\,\text{curl}\,\boldsymbol{j} = -\nabla^2\boldsymbol{H}.$$

This result used in the right-hand side of equation (10.6), together with Stokes's theorem, gives the following expression for the rate of change of the magnetic flux through S_1,

$$\frac{D}{Dt}\int_{S_1} \boldsymbol{H}\,.\,d\boldsymbol{S}_1 = \frac{-c}{\mu\sigma}\oint_{\Gamma_1} \boldsymbol{j}\,.\,d\boldsymbol{r}. \tag{10.7}$$

This equation shows that the change of the magnetic flux passing through a surface fixed to the fluid particles is determined by the electrical resistance. This situation is exactly analogous to that which describes the motion of a vortex in ordinary fluid dynamics with electrical resistance taking the part of viscosity.

Now the momentum equation of ordinary fluid dynamics is obtained if we set $\boldsymbol{H} = 0$ in equation (7.5) and so, remembering that we are neglecting the bulk modulus ζ, we see that

$$\frac{\partial\boldsymbol{v}}{\partial t} = -(\boldsymbol{v}\,.\,\text{grad})\boldsymbol{v} - \frac{1}{\rho}\,\text{grad}\,p + \frac{\eta}{\rho}\,(\nabla^2\boldsymbol{v} + \tfrac{1}{3}\,\text{grad div}\,\boldsymbol{v}). \tag{10.8}$$

The ratio $\mathscr{R}$ of the moduli of the inertia term and the viscous term of this equation is called the **Reynolds number** of the flow and is given by

$$\mathscr{R} = \frac{\rho\,|(\boldsymbol{v}\,.\,\text{grad})\boldsymbol{v}|}{\eta\,|\nabla^2\boldsymbol{v} + \frac{1}{3}\,\text{grad div}\,\boldsymbol{v}|} = \frac{\rho LV}{\eta}, \tag{10.9}$$

where L is a characteristic length and V is a characteristic flow velocity.† The Reynolds number $\mathscr{R}$ is a dimensionless

† The quantity $\nu = \eta/\rho$ is called the **coefficient of kinematic viscosity**.

parameter of considerable importance in ordinary fluid dynamics and is used to characterise conditions of flow and to facilitate comparisons between similar flow configurations. By requiring $\mathscr{R}$ to have the same value in two geometrically similar flow configurations it is possible to select scales of measurement for experiments such that the ratios of the terms in the equations of fluid motion of the two system are the same, thereby enabling direct comparison of the results. Now by comparing equations (5.1) and (10.8) we can similarly define a **magnetic Reynolds number** $\mathscr{R}_m$ as the ratio of the moduli of the convection term and the diffusive term of equation (5.1). We find that

$$\mathscr{R}_m = \frac{|\operatorname{curl}(\boldsymbol{v} \times \boldsymbol{H})|}{\eta_m |\nabla^2 \boldsymbol{H}|} = \frac{LV}{\eta_m}. \tag{10.10}$$

This number is again non-dimensional and is strictly analogous in its properties and uses to the Reynolds number $\mathscr{R}$.

If $\mathscr{R}_m \gg 1$, the electrical resistance of the conductor and consequently the resulting Joule loss and dissipation of the magnetic field are negligible, just as the viscosity can be neglected if $\mathscr{R} \gg 1$. In laboratory experiments using liquid metals such as liquid sodium or mercury $\mathscr{R}_m$ is of the order 10^{-2} to 1 and electrical resistance plays an essential role. In astronomical phenomena where L is extremely large $\mathscr{R}_m$ is of the order 10^6 or more and, moreover, in these cases $\mathscr{R}$ is also sufficiently large so that the fluid may be considered as a perfect fluid of infinite conductivity in which the motion of the fluid is adiabatic. In high temperature plasmas the conductivity is so large that although L is small, $\mathscr{R}_m$ is very large and we encounter a situation similar to that in astrophysics. In what follows a fluid for which all the dissipative effects are negligible, namely η, ζ and χ are equal to zero and σ is infinite, will be called an **ideal conducting fluid**.

If σ is infinite it follows immediately from equation (10.7) that

$$\frac{D}{Dt}\int_{S_1} \boldsymbol{H}.d\boldsymbol{S}_1 = 0. \tag{10.11}$$

Equation (10.11) states that the magnetic flux passing through an arbitrary surface S_1 moving with the fluid is conserved. This theorem was first stated by C. Walén and T. G. Cowling and is sometimes expressed by saying that the magnetic field is **frozen** in the fluid.

Let us now define a **magnetic surface** to be a surface in which magnetic lines of force are embedded and out of which magnetic lines of force do not issue. Then it is clear that the magnetic flux passing through any closed magnetic surface is zero. Alternatively we may consider a **material surface** which is composed of particles of the fluid and, consequently, each element of such a surface moves with the fluid velocity $\boldsymbol{v}(\boldsymbol{x}, t)$. Then, from equation (10.11), it may easily be seen that if at any instant a material surface is a magnetic surface then it is a magnetic surface for all time. This result immediately implies that the magnetic lines of force move with the fluid. Suppose that at $t = 0$ the two material surfaces given by

$$\alpha(\boldsymbol{x}, 0) = c_1, \quad \beta(\boldsymbol{x}, 0) = c_2$$

are also both magnetic surfaces, and that c_1 and c_2 are given constants. Then, at $t = 0$, the intersection of the two surfaces is along a magnetic line of force. The material surface

$$\alpha(\boldsymbol{x}, t) = c_1,$$

which initially coincided with the surface $\alpha(\boldsymbol{x}, 0) = c_1$ and moves with the fluid velocity, is also a magnetic surface for all time; the same being true for another moving surface characterised by $\beta(\boldsymbol{x}, t) = c_2$ which initially coincided with $\beta(\boldsymbol{x}, 0) = c_2$. Hence the intersection of these two moving surfaces always coincides with a magnetic line of force as shown in Fig. 2.

These results will now be used to enable simple derivations of some basic properties of the magnetic field. For example, let us assume that the fluid is incompressible and, consequently, that each closed surface moving with the fluid preserves its volume for all time. In particular, if we think of a volume with a cross-section corresponding to the

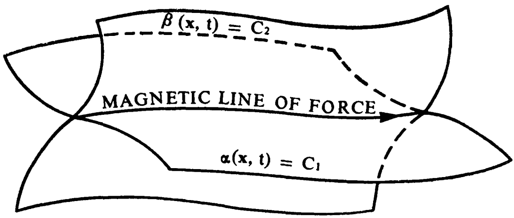

FIG. 2.

area $ABCD$ on the magnetic surface $\alpha(x, t) = c_1$ of Fig. 3 and such that its thickness normal to $\alpha = c_1$ is vanishingly small, this result implies that the cross-sectional area $ABCD$ is invariant as the surface moves and deforms in space. Consequently for a motion in which such a fluid volume elongates in the direction of the magnetic lines of force it follows that the separation AB of the lines of force bounding the volume must decrease as they move to $A'B'$ (see Fig. 3). However, since lines of force can never intersect, the magnetic flux through the end of the volume that is traversed by AB must remain constant, and so it immediately follows that the intensity of the magnetic field increases as the result of such a distortion of the lines of force. This can be expressed in an alternative way by saying that the motion of the fluid element as it stretches itself along the lines of force increases the local magnetic field.

In order to derive a more general property than this we now consider the two further material surfaces specified by

$$\alpha+\delta\alpha = c_1+\delta c_1, \quad \beta+\delta\beta = c_2+\delta c_2$$

which are neighbouring to the material magnetic surfaces

$$\alpha(\boldsymbol{x}, t) = c_1, \quad \beta(\boldsymbol{x}, t) = c_2,$$

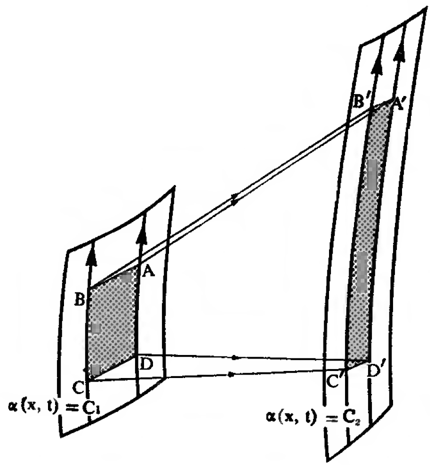

FIG. 3.

respectively. The intersection of these four surfaces defines a narrow rectangular tube and, as we have seen, the lines of intersection of the surfaces all lie along the magnetic lines of force ruling the four surfaces. Consequently the magnetic flux passing through this narrow rectangular tube which moves with the fluid is conserved for all time. At a time t_0 let us consider a point $P_0(\boldsymbol{r}_0, t_0)$ located on one of the four magnetic lines of force determined by the corners of the rectangular tube and introduce a surface element vector $d\boldsymbol{S}_0$ at the point P_0 corresponding to the cross-section of the tube at P_0. Now suppose that at a later time t the point P_0 moves to a point $P(\boldsymbol{r}, t)$ and that a point

$$P_0'(\boldsymbol{r}_0+d\boldsymbol{r}_0,\ t_0),$$

neighbouring to P_0 and located on the same line of force as P_0, moves to a point $P'(\boldsymbol{r}+d\boldsymbol{r},\ t)$, neighbouring to P and

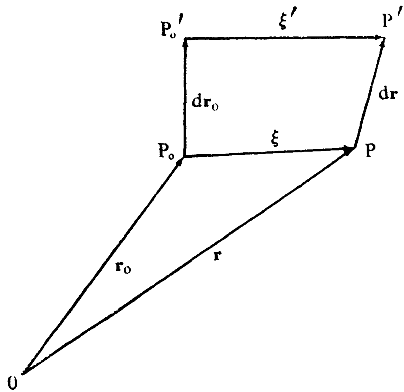

FIG. 4.

located on the same line of force as P. At time t we define the surface element vector $d\boldsymbol{S}$ at P corresponding to the cross-section of the tube at P and parallel to the magnetic field at P.

Denoting the displacement vectors $\overline{P_0P}$ and $\overline{P_0'P'}$ by ξ and ξ', respectively, it is clear from Fig. 4 that

$$d\boldsymbol{r} = d\boldsymbol{r}_0+\xi'-\xi.$$

To see the relationship of $\xi'(\boldsymbol{r}_0+d\boldsymbol{r}_0,\ t_0)$ to $\xi(\boldsymbol{r}_0,\ t_0)$ we need only apply Taylor's theorem to ξ' to obtain

$$\xi'(\boldsymbol{r}_0+d\boldsymbol{r}_0,\ t_0) = \xi(\boldsymbol{r}_0,\ t_0)+(d\boldsymbol{r}_0\,.\,\text{grad})\xi.$$

Combining this result with the previous one then gives the relation

$$d\boldsymbol{r} = d\boldsymbol{r}_0 + (d\boldsymbol{r}_0 . \text{grad})\boldsymbol{\xi}. \qquad (10.12)$$

Now the fact that by construction $\boldsymbol{H}$, $d\boldsymbol{S}$ and $d\boldsymbol{r}$ are all parallel to one another enables equation (10.12) to be rewritten in a more interesting form. To see this let $\hat{\boldsymbol{r}}$ be a unit vector parallel to this triad of vectors and write equation (10.12) in the form

$$dr\hat{\boldsymbol{r}} = dr_0(\hat{\boldsymbol{r}}_0 + (\hat{\boldsymbol{r}}_0 . \text{grad})\boldsymbol{\xi}). \qquad (10.12')$$

Using the frozen-in condition we have

$$\boldsymbol{H} . d\boldsymbol{S} = \boldsymbol{H}_0 . d\boldsymbol{S}_0, \qquad (10.13)$$

whilst from the mass conservation law we have

$$\rho d\boldsymbol{r} . d\boldsymbol{S} = \rho_0 d\boldsymbol{r}_0 . d\boldsymbol{S}_0, \qquad (10.14)$$

from which it follows that

$$\frac{H}{\rho dr} = \frac{H_0}{\rho_0 dr_0}.$$

Using this result in equation (10.12′) to eliminate dr and dr_0 gives

$$\frac{\boldsymbol{H}}{\rho} = \frac{1}{\rho_0}(\boldsymbol{H}_0 + (\boldsymbol{H}_0 . \text{grad})\,\boldsymbol{\xi}). \qquad (10.15)$$

This equation is the general solution of equation (5.1′). If we divide equation (10.15) by $\delta t = t - t_0$ and take the limit as $\delta t \to 0$ we find that

$$\frac{D}{Dt}\left(\frac{\boldsymbol{H}}{\rho}\right) = \frac{1}{\rho}(\boldsymbol{H} . \text{grad})\boldsymbol{v}. \qquad (10.16)$$

The property previously derived in connection with Fig. 3 for an incompressible fluid results immediately from equation (10.15), for when ρ is a constant equal to

ρ_0 it follows at once that $\boldsymbol{H}$ increases when the displacement ξ is such that the fluid stretches itself along the lines of force.

Another property resulting directly from equations (10.12) and (10.15) is that if $d\boldsymbol{r} = d\boldsymbol{r}_0$, so that $(\boldsymbol{H}_0 . \text{grad})\boldsymbol{v} = 0$, then $\boldsymbol{H}/\rho$ is constant along the path of the fluid element. For example, if the magnetic field is in the z-direction and all other quantities are independent of z, then H_z/ρ is constant along each path of a fluid element.

Now the equation of motion (6.3) of the fluid becomes

$$\rho \frac{D\boldsymbol{v}}{Dt} = \boldsymbol{f}^{(m)} + \boldsymbol{f}^{(em)} \tag{10.17}$$

when we assume that $\boldsymbol{f}^{(ex)} = 0$, while for a perfect fluid $\zeta = \eta = 0$ and so, combining equations (6.2) and (7.6) with (10.17) gives,

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\text{grad}\left(p + \frac{\mu \boldsymbol{H}^2}{8\pi}\right) + \frac{\mu}{4\pi}(\boldsymbol{H} . \text{grad})\boldsymbol{H}. \tag{10.18}$$

This equation shows that the force exerted on a fluid element is determined by the **total pressure** p^*, where $p^* = p + \dfrac{\mu \boldsymbol{H}^2}{8\pi}$, and by the magnetic force $\dfrac{\mu}{4\pi}(\boldsymbol{H} . \text{grad})\boldsymbol{H}$.

When, in the case of the magnetic field parallel to the z-axis that we have just examined, there is a functional relationship between H_z and ρ at some initial instant the problem is reducible to the conventional fluid dynamic case and the effect of the magnetic field is simply to modify the compressibility of the fluid.

Analogously to the case of fluid dynamics,† in which the velocity of propagation a of an infinitesimal disturbance (sound speed) is determined by the expression

$$a^2 = \frac{\partial p}{\partial \rho}, \tag{10.19}$$

† See Rutherford, *Fluid Dynamics*, 1959, p. 20.

so the velocity of propagation c^* of an infinitesimal disturbance in this case will be

$$(c^*)^2 = \frac{\partial p^*}{\partial \rho} = a^2+b^2, \tag{10.20}$$

where a is the ordinary speed of sound and where b is defined by the expression

$$b^2 = \frac{\mu}{8\pi}\frac{\partial H_z^2}{\partial \rho}. \tag{10.21}$$

In the simple case in which H_z/ρ is everywhere constant it is easy to see that

$$b = \sqrt{\frac{\mu H_z^2}{4\pi\rho}}. \tag{10.22}$$

This velocity is called the **Alfvén velocity** and will be discussed in the following section.

§ 11. The Alfvén wave. One of the most conspicuous features of magnetohydrodynamics is that even in an incompressible inviscid fluid the basic equations admit solutions representing waves. The existence of such a wave was predicted by H. Alfvén in 1942 and has been confirmed experimentally by Lundquist and Lehnert. The wave has a number of remarkable properties entirely different from those of sound waves and is called the **Alfvén wave.**

Let us assume that initially a fluid at constant pressure is in equilibrium with a uniform magnetic field $\boldsymbol{H}_0$ oriented in the positive z-direction, with $\boldsymbol{H}_0 \doteq H_0\boldsymbol{k}$ where $\boldsymbol{k}$ is the unit vector parallel to the z-axis. Suppose now that the motion of the fluid occurs such that a narrow strip, bounded by two magnetic lines of force L_1 and L_2, is deformed in the manner illustrated in Fig. 5 so that the shaded portion A is transformed into the shaded portion A'.

As was remarked previously, in an incompressible fluid the magnetic field strength will increase as we move from

A to A' due to the stretching of the magnetic lines of force. Moreover, if the total pressure at A' is still balanced by that of the surrounding region, then the force acting on the fluid

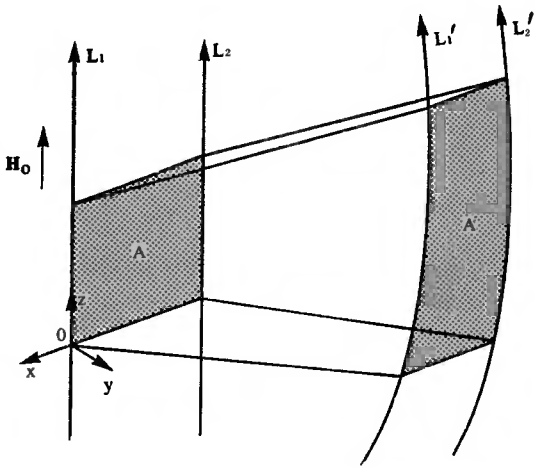

FIG. 5.

in region A' becomes only the **magnetic force** term of equation (10.18) given by

$$\frac{\mu}{4\pi}(\boldsymbol{H}\,.\,\mathrm{grad})\boldsymbol{H}, \tag{11.1}$$

where $\boldsymbol{H}$ is the magnetic field strength at A'. By setting $\boldsymbol{H} = H\boldsymbol{t}$ where $\boldsymbol{t}$ is the unit tangent vector to a magnetic line of force, expression (11.1) may be expanded to give

$$\frac{\mu}{4\pi}(\boldsymbol{H}\,.\,\mathrm{grad})\boldsymbol{H} = \boldsymbol{t}(\boldsymbol{H}\,.\,\mathrm{grad})H + H^2(\boldsymbol{t}\,.\,\mathrm{grad})\boldsymbol{t}.$$

Now the expression $(\boldsymbol{t}\,.\,\mathrm{grad})\boldsymbol{t}$ is simply the rate of change of the tangent vector $\boldsymbol{t}$ along the line of magnetic force and, by simple geometrical considerations,† may be shown to be equal to $\boldsymbol{n}/R$ where $\boldsymbol{n}$ is the principal unit normal to the line of magnetic force and R is its radius of curvature. Accordingly then, we may write

$$\frac{\mu}{4\pi}(\boldsymbol{H}\,.\,\text{grad})\boldsymbol{H} = \frac{\mu}{4\pi}\left\{\boldsymbol{t}(\boldsymbol{H}\,.\,\text{grad})H+\boldsymbol{n}\frac{H^2}{R}\right\}$$

which, since H^2 is always positive, shows that the force acts in a manner to restore the fluid at A' to its original position. So the narrow material strip bounded by the magnetic lines of force, or a magnetic line of force itself, resembles a string subjected to a tension given by formula (11.1) which vibrates around its equilibrium position. Under the assumption that all quantities are independent of x and y and are functions only of z and t the above reasoning may easily be given mathematical expression.

We shall consider displacements only in the y-direction and express the magnetic field and the fluid velocity in the form

$$\boldsymbol{H} = \boldsymbol{H}_0+h(z, t)\boldsymbol{j}$$

$$\boldsymbol{v} = v(z, t)\boldsymbol{j},$$

where $\boldsymbol{j}$ is the unit vector in the y-direction. Then, considering a small displacement $\xi(z, t)$ in the y-direction of a fluid element, by equating forces on the element and retaining only first order terms we may write the following equation of motion,

$$\rho\frac{\partial^2\xi}{\partial t^2} = \frac{\mu H_0}{4\pi}\frac{\partial h}{\partial z}. \tag{11.2}$$

Now, for an incompressible fluid, it follows at once from equation (10.15) that $h = H_0\dfrac{\partial\xi}{\partial z}$ and so equation (11.2)

† See Rutherford, *Vector Methods*, 1954, p. 19.

becomes

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{\mu H_0^2}{4\pi\rho} \frac{\partial^2 \xi}{\partial z^2}. \tag{11.3}$$

This equation is the well-known **wave equation** † and shows that the Alfvén wave is a **transverse** disturbance propagating along the magnetic lines of force with the **Alfvén velocity** $b = \sqrt{\dfrac{\mu H_0^2}{4\pi\rho}}$.

From the expression $h = H_0 \dfrac{\partial \xi}{\partial z}$ and the fact that $v = \dfrac{\partial \xi}{\partial t}$ we at once have that

$$\frac{\partial h}{\partial t} = H_0 \frac{\partial v}{\partial z}, \tag{11.4}$$

while equation (11.3) may be written in the form

$$\frac{\partial v}{\partial t} = \frac{\mu H_0}{4\pi\rho} \frac{\partial h}{\partial z}. \tag{11.5}$$

Multiplying equation (11.5) by $\pm\sqrt{4\pi\rho/\mu}$ and adding it to equation (11.4) then gives

$$\left(\frac{\partial h}{\partial t} \pm b \frac{\partial h}{\partial z}\right) = \pm \sqrt{\frac{4\pi\rho}{\mu}} \left(\frac{\partial v}{\partial t} \pm b \frac{\partial v}{\partial z}\right). \tag{11.6}$$

Now,

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial z} \frac{dz}{dt}$$

which, as the velocity of wave propagation $\dfrac{dz}{dt}$ in the z direction is equal to b, shows that

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + b \frac{\partial h}{\partial z}.$$

† See Coulson, *Waves*, 1949, p. 5.

A similar result is true for $\frac{dv}{dz}$ allowing us to re-write equation (11.6) in the simple form

$$\frac{dh}{dt} = \pm\sqrt{\frac{4\pi\rho}{\mu}}\frac{dv}{dt}.$$

Integrating, we find that

$$h = \pm\sqrt{\frac{4\pi\rho}{\mu}}\,v. \tag{11.7}$$

In laboratory experiments using liquid mercury, because of the small conductivity and the viscosity, such waves attenuate rapidly and are difficult to observe. To illustrate the velocity that may be expected, consider a field of 10^3 gauss, taking $\mu = 1$ and the density of mercury as 13·6 gm/cm^3. Then we see from this last result that the Alfvén velocity would be several kilometres per second. However, when the conductivity is high the waves are readily observable as was the case in a series of elegant experiments on Alfvén waves in plasmas carried out at the Berkeley laboratories in California.

§ 12. Examples.†

1. A vector $\boldsymbol{a}$ is said to be **solenoidal** when div $\boldsymbol{a} = 0$. Prove that when the current vector $\boldsymbol{j}$ is solenoidal all currents must flow in closed loops.

2.†* It is a postulate of the special theory of relativity that for a frame of reference $O'\{x', y', z'\}$ moving with constant velocity V, say, along the x-axis of another frame $O\{x, y, z\}$, the spatial coordinates and times in the two

† Examples 2 and 3 indicate the basic ideas underlying the relativistic transformation laws for the vectors **E** and **B**. For more information reference should be made to: Rindler, *Special Relativity*, 1960.

†* See explanation of asterisk in paragraph five of the Preface.

reference frames are related by the **Lorentz transformation**

$$x' = \frac{x-Vt}{(1-V^2/c^2)^{\frac{1}{2}}},\ y' = y,\ z' = z,\ t' = \frac{t-xV/c^2}{(1-V^2/c^2)^{\frac{1}{2}}}.$$

Denoting differentiation with respect to time in any given system of coordinates by a dot show first that

$$\frac{dt'}{dt} = \frac{1-\dfrac{V}{c^2}\dfrac{dx}{dt}}{(1-V^2/c^2)^{\frac{1}{2}}} = \frac{1-\dfrac{V\dot{x}}{c^2}}{(1-V^2/c^2)^{\frac{1}{2}}}.$$

Hence show that velocities transform according to the laws

$$u'_x = \frac{u_x-V}{1-u_xV/c^2},\ u'_y = \frac{u_y(1-V^2/c^2)^{\frac{1}{2}}}{1-u_xV/c^2},\ u'_z = \frac{u_z(1-V^2/c^2)^{\frac{1}{2}}}{1-u_xV/c^2},$$

where $u_x = \dot{x} = dx/dt$ and $u'_x = \dot{x} = dx'/dt'$ are the x-components of the velocity in the reference frames O and O', respectively; the other components being similarly defined. Show also that the inverse transformations for coordinate variables and velocities may be obtained by interchanging primed and unprimed quantities and by reversing the sign of V. Setting $u^2 = u_x^2+u_y^2+u_z^2$ and using the inverse transformation law for velocities prove that the **Lorentz contraction factor** $(1-u^2/c^2)^{\frac{1}{2}}$ for an object moving with velocity $\boldsymbol{u}$ relative to a given reference frame transforms according to the law

$$(1-u^2/c^2)^{\frac{1}{2}} = \frac{(1-u'^2/c^2)^{\frac{1}{2}}(1-V^2/c^2)^{\frac{1}{2}}}{1+u'_xV/c^2},$$

and hence, since the mass m of a particle moving with velocity $\boldsymbol{u}$ and having a rest mass m_0 is

$$m = m_0/(1-u^2/c^2)^{\frac{1}{2}},$$

that mass transforms according to the transformation

$$m = \frac{m'(1+u'_xV/c^2)}{(1-V^2/c^2)^{\frac{1}{2}}}.$$

3.* It is postulated in relativistic electrodynamics that the Maxwell field equations should have the same form in all sets of coordinates which are in steady motion relative to one another and that the transformations and inverse transformations of electromagnetic quantities should also be symmetrical apart from the sign of the relative velocity of the two frames. The following transformations proposed by Einstein for a vacuum field in which the current vector $\boldsymbol{j}$ has been identified with a charge of density $4\pi q$ moving with velocity $\boldsymbol{u}$ possess all these features:

$$E_x = E'_x,\ E_y = \frac{E'_y + \dfrac{V}{c} B'_z}{(1-V^2/c^2)^{\frac{1}{2}}},\ E_z = \frac{E'_z - \dfrac{V}{c} B'_y}{(1-V^2/c^2)^{\frac{1}{2}}}$$

$$B_x = B'_x,\ B_y = \frac{B'_y - \dfrac{V}{c} E'_z}{(1-V^2/c^2)^{\frac{1}{2}}},\ B_z = \frac{B'_z + \dfrac{V}{c} E'_y}{(1-V^2/c^2)^{\frac{1}{2}}}$$

and

$$q = q' \frac{(1+u'_x V/c^2)}{(1-V^2/c^2)^{\frac{1}{2}}}.$$

Use the results of the previous example to show that

$$\frac{q}{q'} = \left(\frac{1-u'^2/c^2}{1-u^2/c^2}\right)^{\frac{1}{2}},$$

and hence that the electric densities in the two systems are inversely proportional to their Lorentz contraction factors with the result that the total charges e and e' in the two systems are identical.

By defining the force $\boldsymbol{F}$ acting on a particle of mass m moving with velocity $\boldsymbol{u}$ as

$$\boldsymbol{F} = \frac{d}{dt}(m\boldsymbol{u}) = \frac{d}{dt}\left(\frac{m_0 \boldsymbol{u}}{(1-u^2/c^2)^{\frac{1}{2}}}\right),$$

show that the components of force transform according to the law

$$F_x = F'_x + \frac{u'_y V F'_y}{c^2 + u'_x V} + \frac{u'_z V F'_z}{c^2 + u'_x V}$$

$$F_y = \frac{c^2(1 - V^2/c^2)^{\frac{1}{2}} F'_y}{c^2 + u'_x V}$$

$$F_z = \frac{c^2(1 - V^2/c^2)^{\frac{1}{2}} F'_z}{c^2 + u'_x V}.$$

Now, by considering a charge e moving in reference frame O with velocity V along the x-axis so that $u_x = V$, $u_y = u_z = 0$, we have the result that e appears as a stationary charge in reference frame O' and, consequently, that the force acting there has components

$$F'_x = e'E'_x, \quad F'_y = e'E'_y, \quad F'_z = e'E'_z.$$

Hence, since it was established that $e' = e$, use the transformation law for force to prove that

$$F_x = eE_x, \quad F_y = e\left(E_y - \frac{1}{c} u_x B_z\right), \quad F_z = e\left(E_z + \frac{1}{c} u_x B_y\right).$$

Consequently, by allowing the x-axis to be arbitrarily oriented, prove that the Lorentz force $\boldsymbol{F}$ acting on a unit charge moving with velocity $\boldsymbol{u}$ is given by

$$\boldsymbol{F} = \boldsymbol{E} + \frac{1}{c}\, \boldsymbol{u} \times \boldsymbol{B}.$$

4. Show, by making use of the expansions of

$$\text{div}\ [\boldsymbol{H} \times \text{curl}\ \boldsymbol{H}],\ \text{div}\ [\boldsymbol{H} \times (\boldsymbol{v} \times \boldsymbol{H})],$$

or otherwise, that the scalar product of $\boldsymbol{H}$ and the equation for the magnetic field

$$\frac{\partial \boldsymbol{H}}{\partial t} = \text{curl}\,(\boldsymbol{v}\times\boldsymbol{H}) + \frac{c^2}{4\pi\mu\sigma}\nabla^2\boldsymbol{H}$$

may be written in the form

$$\tfrac{1}{2}\frac{\partial \boldsymbol{H}^2}{\partial t} = \boldsymbol{v}\,.\,(\boldsymbol{H}\times\text{curl}\,\boldsymbol{H}) - \frac{4\pi}{\mu\sigma}j^2$$

$$-\text{div}\left[\boldsymbol{H}\times(\boldsymbol{v}\times\boldsymbol{H}) - \frac{c^2}{4\pi\mu\sigma}\boldsymbol{H}\times\text{curl}\,\boldsymbol{H}\right].$$

5. By integrating the final result of Example 4 over a volume V bounded by a fixed surface S show that

$$\frac{\partial}{\partial t}\int_V \frac{\mu \boldsymbol{H}^2}{8\pi}dV = \frac{\mu}{c}\int_V \boldsymbol{v}\,.\,(\boldsymbol{H}\times \boldsymbol{j})dV - \int_V \frac{j^2}{\sigma}dV$$

$$-\frac{\mu}{4\pi}\int_S\left[\boldsymbol{H}\times(\boldsymbol{v}\times\boldsymbol{H}) - \frac{c^2}{4\pi\mu\sigma}\boldsymbol{H}\times\text{curl}\,\boldsymbol{H}\right].d\boldsymbol{S}$$

and hence, when the field is confined to a volume V_1 contained in V, that

$$\frac{\partial}{\partial t}\int_V \frac{\mu \boldsymbol{H}^2}{8\pi}dV = \frac{\mu}{c}\int_V \boldsymbol{v}\,.\,(\boldsymbol{H}\times \boldsymbol{j})dV - \int_V \frac{j^2}{\sigma}dV.$$

6. Using the **summation convention** whereby a repeated suffix in any expression implies summation with respect to that suffix over all the possible values assumed by the suffix, show that the ith component of grad $T_{ik}^{(m)}$, defined to be $\partial T_{ik}^{(m)}/\partial x_k$, is

$$\frac{\partial T_{ik}^{(m)}}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \eta\nabla^2 v_i + (\zeta + \tfrac{1}{3}\eta)\frac{\partial}{\partial x_i}\text{div}\,\boldsymbol{v},$$

where

$$T_{ik}^{(m)} = -p\delta_{ik} + \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \tfrac{2}{3}\delta_{ik}\,\text{div}\,\boldsymbol{v}\right) + \zeta\delta_{ik}\,\text{div}\,\boldsymbol{v}.$$

Hence show that the equation of motion for the flow of a viscous electrically-conducting fluid is

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\text{grad } p + \frac{\mu}{c}\boldsymbol{j}\times\boldsymbol{H}+\eta\nabla^2\boldsymbol{v}$$

$$+(\zeta+\tfrac{1}{3}\eta) \text{ grad div } \boldsymbol{v}+\boldsymbol{f}^{(ex)}.$$

7. The total energy U per unit volume occupied by a fluid is given by the expression $U = \frac{1}{2}\rho\boldsymbol{v}^2 + \frac{\mu\boldsymbol{H}^2}{8\pi} + \rho e$. Show that U satisfies the energy conservation law

$$\frac{\partial U}{\partial t} = -\text{div } \boldsymbol{g},$$

where the energy flow vector $\boldsymbol{g}$ is given by

$$\boldsymbol{g} = \rho\boldsymbol{v}(\tfrac{1}{2}\boldsymbol{v}^2+e+p\tau)+\frac{\mu}{4\pi}\left\{\boldsymbol{H}\times(\boldsymbol{v}\times\boldsymbol{H})-\frac{c}{4\pi\mu\sigma}\boldsymbol{H}\times\text{curl }\boldsymbol{H}\right\}$$

$$-\eta\sum_{i,k=1}^{3}\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}-\tfrac{2}{3}\delta_{ik}\text{ div }\boldsymbol{v}\right)v_i\boldsymbol{e}_k-\chi\text{ grad } T,$$

the notation being that of § 8.

8. Defining the flux ϕ of a vector $\boldsymbol{Q}$ through a surface S bounded by a closed curve Γ moving with velocity $\boldsymbol{v}(\boldsymbol{x}, t)$ to be

$$\phi = \int_S \boldsymbol{Q}\,.\,d\boldsymbol{S},$$

prove, by arguments similar to those used in § 10, the following integral rate of change theorem

$$\frac{D\phi}{Dt} = \int_S\left[\frac{\partial \boldsymbol{Q}}{\partial t}+\boldsymbol{v}\text{ div }\boldsymbol{Q}+\text{curl}(\boldsymbol{Q}\times\boldsymbol{v})\right].\,d\boldsymbol{S}.$$

Then, by identifying $\boldsymbol{Q}$ with the magnetic field vector $\boldsymbol{H}$, obtain the "frozen in" condition for a perfectly conducting fluid.

9. Show by combining the magnetic field equation for a perfectly conducting inviscid fluid,

$$\frac{\partial \boldsymbol{H}}{\partial t} = \operatorname{curl}(\boldsymbol{v} \times \boldsymbol{H}),$$

and the equation of continuity

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}) = 0,$$

that

$$\frac{D}{Dt}\left(\frac{\boldsymbol{H}}{\rho}\right) = \frac{1}{\rho}(\boldsymbol{H}.\operatorname{grad})\boldsymbol{v}. \qquad (A)$$

Then, defining the **vorticity** $\boldsymbol{\omega}$ by the relationship $\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{v}$ and using the Cartesian coordinates $O\{x_1, x_2, x_2\}$ with the unit vector $\boldsymbol{e}_i$ associated with x_i, make use of the decomposition

$$\frac{1}{2\rho}\sum_{k=1}^{3}\left\{H_k\left(\frac{\partial v_i}{\partial x_k} - \frac{\partial v_k}{\partial x_i}\right) + H_k\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)\right\}$$

of the ith component of the right-hand side of (A) to prove that

$$\frac{D}{Dt}\left(\frac{\boldsymbol{H}}{\rho}\right) = \frac{1}{2\rho}\left\{\boldsymbol{\omega} \times \boldsymbol{H} + \sum_{i,k=1}^{3} S_{ik} H_k \boldsymbol{e}_i\right\},$$

wheie $S_{ik} = \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)$ is called the deformation tensor of the fluid.

10. Show that for an incompressible ideal fluid of density ρ_0 in a conservative force field with scalar potential ψ (acceleration $\boldsymbol{a}$ experienced by a unit mass is $\boldsymbol{a} = -\operatorname{grad} \psi$) the equation of fluid motion takes the form

$$\rho_0 \frac{D\boldsymbol{v}}{Dt} = -\operatorname{grad}\left(p + \frac{\mu H^2}{8\pi} + \rho_0 \psi\right) + \frac{\mu}{4\pi}(\boldsymbol{H}.\operatorname{grad})\boldsymbol{H}.$$

Assume that the fluid is subjected to a uniform magnetic field $\boldsymbol{H}_0$, and orient the coordinate axes so that $\boldsymbol{H}_0 = H_0\boldsymbol{k}$, where $\boldsymbol{k}$ is the unit vector in the z-direction. Show that if a small disturbance producing a fluid velocity $\boldsymbol{v}$ results in the magnetic field becoming $\boldsymbol{H} = \boldsymbol{H}_0+\boldsymbol{h}$, then the equations for the magnetic field, for the fluid motion and for mass conservation take the form

$$\frac{\partial \boldsymbol{h}}{\partial t} = H_0 \frac{\partial \boldsymbol{v}}{\partial z},$$

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\frac{1}{\rho_0} \operatorname{grad}\left(p+\frac{\mu \boldsymbol{H}_0 \cdot \boldsymbol{h}}{4\pi}+\rho_0\psi+\frac{\mu H_0^2}{8\pi}\right)+\frac{\mu H_0}{4\pi\rho_0}\frac{\partial \boldsymbol{h}}{\partial z}$$

and

$$\operatorname{div} \boldsymbol{v} = 0,$$

when terms of the order of $\boldsymbol{h}^2$, $\boldsymbol{h}\cdot\boldsymbol{v}$ and $\boldsymbol{v}^2$ are neglected. By using the results $\operatorname{div} \boldsymbol{v} = \operatorname{div} \boldsymbol{H} = \operatorname{div} \boldsymbol{h} = 0$, show that

$$U = p+\frac{\mu \boldsymbol{H}_0 \cdot \boldsymbol{h}}{4\pi}+\rho_0\psi+\frac{\mu H_0^2}{8\pi}$$

is a harmonic function. Hence show that since U is harmonic and bounded everywhere and has no singularities it must be a constant.† Finally, make use of this result to show that the equations for $\boldsymbol{v}$ and $\boldsymbol{h}$ simplify to the following equations

$$\boldsymbol{v} = \pm\sqrt{\frac{\mu}{4\pi\rho_0}}\,\boldsymbol{h},$$

$$\frac{\partial^2 \boldsymbol{h}}{\partial t^2} = \frac{\mu H_0^2}{4\pi\rho_0}\frac{\partial^2 \boldsymbol{v}}{\partial z^2},$$

$$\frac{\partial^2 \boldsymbol{v}}{\partial t^2} = \frac{\mu H_0^2}{4\pi\rho_0}\frac{\partial^2 \boldsymbol{v}}{\partial z^2},$$

† See, for example, the theorem on p. 111 of Phillips, *Functions of a Complex Variable*, 1958.

describing the propagation of a general Alfvén wave in which the small disturbance vectors $\boldsymbol{h}$ and $\boldsymbol{v}$ travel with the Alfvén speed $b = \sqrt{\dfrac{\mu H_0^2}{4\pi\rho_0}}$.

11. By definining the vorticity $\boldsymbol{\omega}$ by the relation $\boldsymbol{\omega} = \text{curl } \boldsymbol{v}$, prove that in an incompressible conducting fluid of density ρ_0 subject to a uniform magnetic field $\boldsymbol{H}_0$, both $\boldsymbol{\omega}$ and the current $\boldsymbol{j}$ propagate with the Alfvén speed. Prove also that

$$j = \pm c \sqrt{\frac{\rho}{4\pi\mu}}\, \omega.$$

CHAPTER II

MAGNETOHYDRODYNAMIC BOUNDARY CONDITIONS

§ 13. **General considerations.** Problems are frequently encountered in magnetohydrodynamics in which either a physical boundary or an internal interface between fluids occurs across which different solutions must be joined. In such circumstances the physical situation is idealised by supposing a geometrical surface to exist between the two regions in question across which the field variables are related by laws which allow special kinds of discontinuities to take place in certain field variables when crossing the surface.

Although the notion of a discontinuity in the field variables is intuitively obvious it is nevertheless worth while defining these ideas a little more precisely. The most obvious type of dicontinuity occurs when an actual jump takes place in the value of a field variable when crossing a discontinuity surface D. Thus, if P_1 and P_2 are points on opposite sides of D, we define the jump $\Delta \boldsymbol{F}$ in $\boldsymbol{F}$ say, at a point P of D to be

$$\Delta \boldsymbol{F} = \lim_{P_1, P_2 \to P} \{\boldsymbol{F}(P_1) - \boldsymbol{F}(P_2)\}. \tag{13.1}$$

This type of jump is a simple discontinuity in $\boldsymbol{F}$ and if we require that $\Delta \boldsymbol{F} \to 0$ as P_1, $P_2 \to P$ we arrive at the usual form of the definition of a continuous vector function $\boldsymbol{F}$ of several variables. A similar definition applies to a scalar function F as may be seen by considering the scalar components of (13.1). We shall see that discontinuities of this type, frequently called **strong discontinuities**, occur in the theory of magnetohydrodynamic shocks.

A weaker form of discontinuity in which the functional value of $\boldsymbol{F}$ itself is continuous across D but where some derivative of $\boldsymbol{F}$ may be discontinuous is also of very real interest. Discontinuities of this type, frequently called **weak discontinuities** to distinguish them from the shock-like jump discontinuities, will be of fundamental importance when we consider the theory of characteristics and its applications.

We remark that one form of strong discontinuity that is implied by definition (13.1) is that for a vector function $\boldsymbol{F}$, the normal component of $\boldsymbol{F}$ may be continuous across D whilst the tangential component is discontinuous, or vice versa. This is a form of strong discontinuity that often occurs in boundary conditions.

We may, for example, be considering the interface between a fluid and a moving insulating wall, or perhaps the interface between a fluid and a vacuum. In the first case, an obvious requirement on the velocity vector $\boldsymbol{v}$ of the fluid and the velocity vector $\boldsymbol{u}$ of the insulating wall is that if the fluid is to remain continuous up to the wall without the appearance of voids, and hence **cavitation** is not to take place, they must always have identical normal components of velocity. Denoting the normal to the wall by $\boldsymbol{n}$, this fluid boundary condition can be expressed mathematically by the requirement that

$$\boldsymbol{n}.(\boldsymbol{v}-\boldsymbol{u}) = 0 \qquad (13.2)$$

at all points of the wall. Also, if the fluid is viscous and non-cavitating, we know from observation that the fluid will stick to the wall and so for a viscous fluid we require a stronger boundary condition that (13.2); namely that $\boldsymbol{v} = \boldsymbol{u}$ at all points of the wall. Obviously condition (13.2) is also the condition that must be satisfied at an internal fluid-fluid interface in a non-viscous fluid.

The nature of the electromagnetic boundary conditions to be applied at discontinuity surfaces is less obvious, as is also the case in the second example, for there, although the fluid domain is bounded by a vacuum, this need not be the case for the electromagnetic field variables.

We shall examine the fluid and electromagnetic discontinuities occurring in magnetohydrodynamic shocks when we discuss shock phenomena and so now we shall only discuss the discontinuities experienced by the electromagnetic field variables at interfaces and boundaries.

So far the form in which we have displayed the electromagnetic field equations has been appropriate only when the field variables have been differentiable. At any points of discontinuity of the field variables, such as the boundary between a conducting wall and a fluid, this assumption of differentiability is violated and the electromagnetic differential equations cease to be valid. To overcome this difficulty it will be necessary to re-formulate the relevant equations in a form which permits such discontinuities to occur and then, by considering these equations in regions immediately adjacent to the boundary, to obtain boundary conditions which are compatible with the solution elsewhere in the fluid. Since integrals involving discontinuous functions present no difficulty we shall in fact transform certain of the pre-Maxwell equations into their so-called **integral form** when, as we shall show, the results may be used directly to obtain the desired boundary conditions.

First, though, we must take note of the fact that the electromagnetic field variables in magnetohydrodynamics are governed by the pre-Maxwell equations and not by the full Maxwell equations from which it is customary to deduce boundary conditions for classical electromagnetic phenomena. Pausing then for a moment to examine the pre-Maxwell equations, we see that equation (2.2) is of fundamental importance in that it provides both the link between the vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ and the only mechanism by which time variation may enter explicitly. Furthermore, an examination of the summarised electromagnetic field equations of magnetohydrodynamics contained in § 9 shows that it is $\boldsymbol{H}$ rather than $\boldsymbol{E}$ that is fundamental. This follows from the fact that when $\boldsymbol{H}$ and $\boldsymbol{v}$ have been determined from the combined electromagnetic and fluid field equations, then the electric field vector, the charge and the current may be immediately obtained from equations (3.10), (4.3) and (2.9).

One of the differences between the pre-Maxwell equations and the Maxwell equations has already been demonstrated in § 2, when it was shown that a surprising consequence of the pre-Maxwell equations was that no electric charge can exist in a conductor which obeys Ohm's law and is at rest. There are indeed other consequences of these equations which result in the familiar boundary conditions of classical electromagnetic phenomena not being entirely appropriate as boundary conditions for the pre-Maxwell equations. Their deficiency stems from the fact that the field variables in the pre-Maxwell equations are linked in a rather different way from those in the Maxwell equations. The result of this is that although physically it may often seem reasonable that the pre-Maxwell equations should approximate the Maxwell equations, mathematically, when classical electromagnetic boundary conditions are imposed on the pre-Maxwell equations the problem can become overspecified. The consequences of the over-specification are generally that a natural conservation law will be disobeyed or that a rapid transient behaviour will be inaccurately described since then condition (I) will be violated.

§ 14. Integral form of the pre-Maxwell equations. Since the discontinuity surfaces which will concern us may be either stationary or moving we shall derive the integral form of the pre-Maxwell equations in a form which is appropriate to either type of surface. We begin then by considering the total rate of change with respect to time of the surface integral of $\boldsymbol{H}$ over a moving geometrical surface $S_1(t)$, with vector surface element $d\boldsymbol{S}_1$, moving with velocity $\boldsymbol{u}$ and bounded by a moving piecewise smooth closed curve $\Gamma_1(t)$ with vector line element $d\boldsymbol{r}$. Thus, in physical terms, we are seeking the mathematical form taken by **Faraday's law of induction** † when defining the electromotive force (e.m.f.) in terms of the rate of change of the magnetic flux through a moving and deforming surface.

To do this we apply the theorem expressed in Example 8 of § 12, to the magnetic flux $\phi(t)$ defined, as in equation (10.1), by

$$\phi(t) = \int_{S_1(t)} \boldsymbol{H} . d\boldsymbol{S}_1$$

to obtain

$$\frac{D}{Dt}\int_{S_1(t)} \boldsymbol{H} . d\boldsymbol{S}_1 = \int_{S_1(t)} \left\{\frac{\partial \boldsymbol{H}}{\partial t} + \boldsymbol{u} \operatorname{div} \boldsymbol{H} + \operatorname{curl}(\boldsymbol{H}\times\boldsymbol{u})\right\} . d\boldsymbol{S}_1. \tag{14.1}$$

If, now, we make use of the pre-Maxwell equations (2.2) and (2.3) this becomes

$$\frac{D}{Dt}\int_{S_1(t)} \boldsymbol{H} . d\boldsymbol{S}_1 = -\frac{c}{\mu}\int_{S_1(t)} \operatorname{curl}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{u}\times\boldsymbol{B}\right) . d\boldsymbol{S}_1. \tag{14.2}$$

Applying Stokes's theorem to equation (14.2) then gives us the integral form of the pre-Maxwell equation (2.2) which is in fact simply the desired statement of Faraday's law of induction for a moving surface, namely,

$$\frac{D}{Dt}\int_{S_1(t)} \boldsymbol{H} . d\boldsymbol{S}_1 = -\frac{c}{\mu}\oint_{\Gamma_1(t)} \left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{u}\times\boldsymbol{B}\right) . d\boldsymbol{r}. \tag{14.3}$$

† See Coulson, *Electricity*, 1951, §§ 91 and 93.

When $\boldsymbol{u} = \boldsymbol{0}$ this reduces to the usual expression of Faraday's law of induction

$$\text{e.m.f.} = \oint_{\Gamma_1} \boldsymbol{E}.d\boldsymbol{r} = -\frac{1}{c}\frac{\partial}{\partial t}\int_{S_1} \boldsymbol{B}.d\boldsymbol{S}_1. \tag{14.4}$$

We shall also require the integral form of equation (2.3) expressing the solenoidal property of the magnetic induction vector $\boldsymbol{B}$. If we consider a volume $V(t)$ with volume element dV bounded by the closed moving surface $S(t)$ with surface element $d\boldsymbol{S}$, then we may write

$$\int_{V(t)} \text{div}\, \boldsymbol{B}\, dV = 0. \tag{14.5}$$

Applying the Gauss divergence theorem to equation (14.5) then gives us the desired integral form of pre-Maxwell equation (2.3),

$$\int_{S(t)} \boldsymbol{B}.d\boldsymbol{S} = 0. \tag{14.6}$$

It is from these two equations that we shall now obtain the compatible boundary conditions for a general moving discontinuity surface.

§ 15. Electromagnetic boundary conditions. To see how equations (14.3) and (14.6) enable us to determine the discontinuity conditions across a discontinuity surface we now proceed as follows. Consider a moving surface $D(t)$ across which the field variables may possess discontinuities and draw an arbitrary infinitesimal plane rectangular loop $\Gamma_1(t)$ around a point P of $D(t)$ in such a manner that the normal $\boldsymbol{b}$ to the plane of the loop $\Gamma_1(t)$ is tangent to the surface $D(t)$ at P. Then, denoting the tangent vector to $D(t)$ which is normal to $\boldsymbol{b}$ by $\boldsymbol{t}$ (see Fig. 6); the normal $\boldsymbol{n}$ to $D(t)$ lies in the plane of $\Gamma_1(t)$ and is given by the relation $\boldsymbol{n} = \boldsymbol{t} \times \boldsymbol{b}$. Here $\boldsymbol{t}$ and $\boldsymbol{b}$ may be any orthogonal unit vectors in the tangent plane at P, but of course $\boldsymbol{n}$ is unique apart from sign. We shall denote the region into which $\boldsymbol{n}$ points by $(+)$ and the region on the other side of $D(t)$ by $(-)$. The loop $\Gamma_1(t)$ is assumed to be oriented in such a manner that its short sides of length δ are parallel to $\boldsymbol{n}$ and its long sides of length l are parallel to $\boldsymbol{t}$.

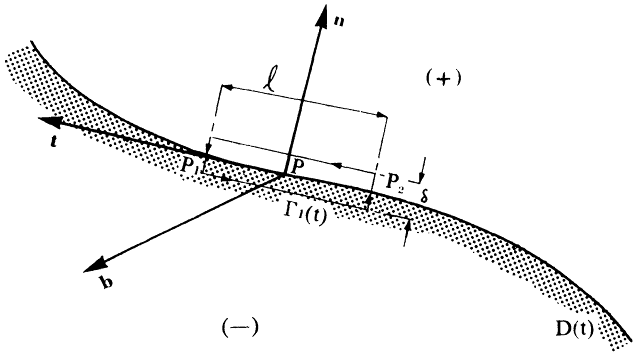

FIG. 6.

We now identify the arbitrary piecewise continuous loop $\Gamma_1(t)$ of equation (14.3) with the small rectangular loop $\Gamma_1(t)$ that we have just defined and the $S_1(t)$ with the small plane area contained by $\Gamma_1(t)$. Furthermore we shall suppose that $\Gamma_1(t)$ always moves with the discontinuity surface $D(t)$ in such a manner that it preserves its orientation. Vector $\boldsymbol{u}$ becomes the velocity of the discontinuity surface $D(t)$. Under these conditions equation (14.3) can then be approximated by the expression

$$\frac{D}{Dt}(\boldsymbol{H}.\boldsymbol{b})\delta l = \frac{c}{\mu}\left[\boldsymbol{E}+\frac{1}{c}\boldsymbol{u}\times\boldsymbol{B}\right].\boldsymbol{t}l$$

$$-\frac{c}{\mu}\left\{\left(\overline{\boldsymbol{E}+\frac{1}{c}\boldsymbol{u}\times\boldsymbol{B}}\right)_{P_2}.\boldsymbol{n}-\left(\overline{\boldsymbol{E}+\frac{1}{c}\boldsymbol{u}\times\boldsymbol{B}}\right)_{P_1}.\boldsymbol{n}\right\}\delta, \quad (15.1)$$

where $[X] \equiv X^+ - X^-$ signifies the jump in quantity X

across $D(t)$, and where $(\bar{Y})_{P_i}$ signifies the average value of Y at P_i. So, dividing by l and letting $\delta \to 0$, we find that provided $\boldsymbol{E}$, $\boldsymbol{B}$ and $\dfrac{D}{Dt}(\boldsymbol{H}.\boldsymbol{b})$ remain bounded,

$$\left[\boldsymbol{E} + \frac{1}{c}\boldsymbol{u} \times \boldsymbol{B}\right].\boldsymbol{t} = 0. \tag{15.2}$$

As $\boldsymbol{t}$ is an arbitrary tangent vector at P it follows at once that

$$\left[\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{u} \times \boldsymbol{B}\right)_t\right] = 0, \tag{15.3}$$

where the suffix t denotes the tangential component. In words this says that the tangential component of the vector $\boldsymbol{E} + \dfrac{1}{c}\boldsymbol{u} \times \boldsymbol{B}$ is continuous across the discontinuity surface $D(t)$. When $\boldsymbol{u} = \boldsymbol{0}$ this reduces to the familiar statement † that the tangential component of the electric field is continuous when crossing a fixed discontinuity surface.

It now only remains to use equation (14.6) to deduce the discontinuity conditions for $\boldsymbol{B}$ normal to $D(t)$. To do this we identify the surface $S(t)$ of equation (14.6) with the surface of an arbitrarily small cylindrical volume $V(t)$ which is constructed so that it encloses a small part of the discontinuity surface $D(t)$ and always moves with $D(t)$. We shall suppose that the axis of this cylinder is normal to a point P of $D(t)$ and that the end faces are parallel to the tangent plane to $D(t)$ at P.

Equation (14.6) may then be approximated by the expression

$$0 = \int_{S(t)} \boldsymbol{B}.d\boldsymbol{S} = \boldsymbol{B}^{(+)}.d\boldsymbol{S}^{(+)} + \boldsymbol{B}^{(-)}.d\boldsymbol{S}^{(-)}$$
$$+ \text{contributions from the sides of the cylinder} \tag{15.4}$$

† See Coulson, *loc. cit.*, p. 58.

where, again, (+) and (−) signify values on opposite sides of $D(t)$. Now, since $d\boldsymbol{S}^{(+)} = -d\boldsymbol{S}^{(-)} = \boldsymbol{n}dS$, where $\boldsymbol{n}$ is the normal to $D(t)$ at P and dS is the area of the cylinder end, as the cylinder height shrinks to zero equation (15.4) becomes

$$[\boldsymbol{n}.\boldsymbol{B}]dS = 0. \tag{15.5}$$

Because the cylinder was arbitrarily located on $D(t)$, and had an arbitrary cross-sectional area dS, this result at once implies the following boundary condition on $\boldsymbol{B}$, namely

$$[\boldsymbol{n}.\boldsymbol{B}] = 0. \tag{15.6}$$

Following the notation that is often used, whereby the normal component of a vector is denoted by the suffix n, this result simply says in physical terms that the normal component B_n of vector $\boldsymbol{B}$ is continuous across a discontinuity surface, or that

$$[B_n] = 0. \tag{15.6'}$$

When, for some reason, the normal component of $\boldsymbol{B}$ vanishes everywhere along the discontinuity surface the electric boundary condition (15.2) can be slightly simplified as follows. Writing the vector $\boldsymbol{E}+\frac{1}{c}\boldsymbol{u}\times\boldsymbol{B}$ as the sum of tangential and normal components, and using result (15.3), it is easily seen that the electric boundary condition is equivalent to

$$\boldsymbol{n}\times[\boldsymbol{E}+\frac{1}{c}\boldsymbol{u}\times\boldsymbol{B}] = \boldsymbol{0}.$$

Expanding the triple vector product in this equation and using the condition $\boldsymbol{n}.\boldsymbol{B} = B_n = 0$ then gives

$$\boldsymbol{n}\times[\boldsymbol{E}] = \frac{1}{c}[(\boldsymbol{n}.\boldsymbol{u})\boldsymbol{B}].$$

However, it follows from equation (13.2) that $\boldsymbol{n}.\boldsymbol{u}$ is continuous across the discontinuity surface and so this may finally be written in the form

$$\boldsymbol{n}\times[\boldsymbol{E}] = \frac{1}{c}(\boldsymbol{n}\,.\,\boldsymbol{u})[\boldsymbol{B}]. \tag{15.7}$$

Let us again consider Fig. 6 and this time form the line integral of $\boldsymbol{H}$ around Γ_1 when, by using Stokes's theorem and equation (2.9), we find that

$$\oint_{\Gamma_1} \boldsymbol{H}\,.\,d\boldsymbol{r} = \frac{4\pi}{c}\int_{S_1} \boldsymbol{j}\,.\,d\boldsymbol{S}_1.$$

By essentially the same arguments as were used in deriving the electric boundary condition we then readily arrive at the result

$$\boldsymbol{n}\times[\boldsymbol{H}] = \frac{4\pi}{c}\lim_{\delta\to 0}(\boldsymbol{j}\delta).$$

When the conductivities on adjacent sides of the discontinuity surface are finite $\boldsymbol{j}$ remains finite and so, in the limit as $\delta\to 0$, we obtain the boundary condition

$$\boldsymbol{n}\times[\boldsymbol{H}] = \boldsymbol{0}. \tag{15.8}$$

However, when one of the regions adjacent to the discontinuity surface is a perfect conductor, $\boldsymbol{j}$ may possess an infinity such that $\boldsymbol{j}_S = \lim_{\delta\to 0}(\boldsymbol{j}\delta)$ is finite and defines the density of the **surface distribution of current**. The general discontinuity condition for the tangential component of $\boldsymbol{H}$ thus becomes

$$\boldsymbol{n}\times[\boldsymbol{H}] = \frac{4\pi}{c}\boldsymbol{j}_S. \tag{15.9}$$

Special cases arise when one of the regions adjacent to a discontinuity surface is either a **perfect conductor** ($\sigma = \infty$) or a **perfect insulator** ($\sigma = 0$). The case of the perfect conductor is particularly simple for then, instead of it being necessary to obtain solutions in both the perfect conductor and the fluid which must be joined in a suitable manner across the interface, the solution in the perfectly conducting region may be ignored and replaced by a simple boundary condition to be obeyed by the fluid on the physical boundaries of the conductor. For a perfect conductor moving with velocity $\boldsymbol{w}$, say, **Ohm's law** which is expressed by equation (3.10) takes the form

$$\boldsymbol{E}+\frac{1}{c}\boldsymbol{w}\times\boldsymbol{B}=\boldsymbol{0}. \tag{15.10}$$

It is obvious from this result that a perfect conductor is characterised by the physical requirement that there should be no tangential electric field experienced at the surface with respect to a reference frame moving with the conductor. A consequence of this result which is implied by equation (15.2) is that the tangential surface component of the electric field in the fluid must vanish. Accordingly, at the boundary of a fluid moving with velocity $\boldsymbol{v}$ and a perfect conductor moving with velocity $\boldsymbol{u}$, we must clearly have the electric boundary condition

$$\boldsymbol{n}\times\left(\boldsymbol{E}+\frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right)=\boldsymbol{0} \tag{15.11}$$

and the fluid boundary condition

$$\boldsymbol{n}.\boldsymbol{v}=\boldsymbol{n}.\boldsymbol{u}. \tag{15.12}$$

Now, referring back for the moment to Faraday's law of induction (14.3), we see that the velocity $\boldsymbol{u}$ of the discontinuity surface $\Gamma_1(t)$ appears only in the form of the triple scalar product $\boldsymbol{u}\times\boldsymbol{B}.d\boldsymbol{r} = d\boldsymbol{r}\times\boldsymbol{u}.\boldsymbol{B}$, thus showing that it is in fact only the component of $\boldsymbol{u}$ that is normal to $\Gamma_1(t)$ that really appears in the result. Because of this conclusion and our choice of the limiting form of $\Gamma_1(t)$ when deriving the electric boundary condition, we may conclude that equation (15.11) depends on $\boldsymbol{n}.\boldsymbol{v}$ only. Expanding the triple vector product in equation (15.11) and examining the terms then shows that the compatible boundary condition for B_n, in order that the result is independent of $\boldsymbol{n}.\boldsymbol{v}$, must be

$$B_n = 0. \tag{15.13}$$

We shall use this as the **magnetic boundary condition** on a perfect conductor. A brief discussion of the form of the boundary conditions when two perfectly conducting regions are adjacent to one another is contained in § 36 (iii).

Condition (15.13), when used in conjunction with equation (15.11) and the fluid boundary condition (15.12) which is true for all non-cavitating fluids, allows us to replace the fluid velocity $\boldsymbol{v}$ in electric boundary condition (15.11) by the velocity $\boldsymbol{u}$ of the boundary itself. This gives as the **electric boundary condition** for a moving perfect conductor,

$$\boldsymbol{n} \times \left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{u} \times \boldsymbol{B}\right) = \boldsymbol{0}. \tag{15.14}$$

The use of these boundary conditions may be illustrated by an example of considerable physical importance called the **magnetohydrodynamic free boundary problem.** This problem occurs in studies of high temperature plasma containment by an electromagnetic field. In an idealised form, it may be considered to involve study of the motion of a perfectly conducting fluid, in which $\boldsymbol{E} = \boldsymbol{B} = \boldsymbol{0}$, that is bounded by a vacuum region in which there is an electromagnetic field. When the interface has a surface charge the electromagnetic field suffers an abrupt change across the boundary and, if the configuration is stable, the plasma may be confined within a region purely by means of the electromagnetic field.

The vacuum region surrounding such an electromagnetically confined plasma may itself usually be considered to be bounded by a rigid perfectly conducting wall. The appropriate boundary condition on the wall may be derived from equation (15.14) and is

$$\boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{0}. \tag{15.15}$$

The electric boundary condition here, which expresses the fact that $\boldsymbol{E}$ is normal to the surface of the perfectly conducting rigid wall, when taken in conjunction with equation (2.2), immediately implies that on the wall we also have

$$\boldsymbol{n} \cdot \frac{\partial \boldsymbol{B}}{\partial t} = 0. \tag{15.16}$$

When insulating regions occur within the flow pattern of a conducting fluid it is not usually possible to replace the combined problems of finding the solution within the insulators and the solution in the fluid by simply specifying boundary conditions on the physical extremities of the insulators. It is easy to see from the summary of the electromagnetic equations presented in § 9 that in an insulating region we must require $\boldsymbol{B}$ to satisfy the equations

$$\text{curl}\, \boldsymbol{B} = \boldsymbol{0}, \quad \text{div}\, \boldsymbol{B} = 0 \tag{15.17}$$

and, in general, it is this solution that must be joined by jump conditions to the solution in the fluid. However, since our applications involving insulating boundaries will be rather simple, these problems will not arise and so we shall not examine this further.

At the surface of an insulator which is penetrated by the magnetic induction vector $\boldsymbol{B}$ we shall take as our magnetic boundary condition the requirement that the vector $\boldsymbol{B}$ itself should be continuous across the boundary, that is that

$$[\boldsymbol{B}] = \boldsymbol{0}. \tag{15.18}$$

To see that this condition implies that the normal component $j_n = \boldsymbol{n} \cdot \boldsymbol{j}$ of the current vector vanishes at the surface of an insulator, as would be expected from physical considerations, we proceed as follows. Using equation (2.9) and adopting the orthogonal coordinates $O\{x, y, z\}$ at an arbitrary point of the insulating surface with the z-axis normal to that surface, we find that the jump in the normal component j_n of $\boldsymbol{j}$ across a boundary is

$$[j_n] = \frac{c}{4\pi\mu}\left[\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right].$$

However, equation (15.18) implies that $[\boldsymbol{B}_t] = 0$, from which it immediately follows that

$$[j_n] = 0. \tag{15.19}$$

Since $\boldsymbol{j} = \boldsymbol{0}$ inside an insulator equation (15.19) reduces to the obvious physical fact that $j_n = 0$ on the surface of an insulator.

CHAPTER III

INCOMPRESSIBLE MAGNETOHYDRODYNAMIC FLOW

§ 16. The equations of incompressible magnetohydrodynamic flow. We have already encountered one important example of incompressible magnetohydrodynamic flow when, in the first Chapter, we demonstrated the possibility of Alfvén waves occurring in an incompressible perfectly conducting inviscid fluid. This very special case of wave motion will be examined in more detail in Chapter V when more general types of magnetohydrodynamic waves will be studied and some of their fundamental properties will be derived. However, before proceeding with a general study of waves in a perfectly conducting inviscid fluid, let us first examine some of the simpler types of flow that can occur in a fluid which is both incompressible and viscous.

Two distinct cases of incompressible flow should now be clearly distinguished and to do this we must consider the continuity equation (6.5) which, since ρ is independent of t, becomes

$$(\boldsymbol{v}\,.\,\mathrm{grad})\rho+\rho\,\mathrm{div}\,\boldsymbol{v} = 0. \qquad (6.5')$$

When ρ is homogeneous, and so has the same constant value throughout the fluid, this reduces to the familiar **incompressibility condition**

$$\mathrm{div}\,\boldsymbol{v} = 0. \qquad (16.1)$$

If, now, we define a **streamline** to be a line in the fluid with the property that any tangent to a point on the streamline determines the fluid direction of motion at that point then, when ρ is constant along a streamline, we must have

$$(\boldsymbol{v}.\text{grad})\rho = 0. \tag{16.2}$$

Using this result in equation (6.5′) shows that when ρ is constant along a streamline we again obtain div $\boldsymbol{v} = 0$. The latter case occurs when the density ρ has different but constant values along different streamlines. The former occurs when ρ is homogeneous and so has the same constant value throughout the entire fluid.

If we assume that there are no external forces acting, and hence that $\boldsymbol{f}^{(ex)} = \boldsymbol{0}$ then, using equation (16.1), the equation of motion (7.5) becomes

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\text{grad}\, p + \frac{\mu}{c}\boldsymbol{j}\times\boldsymbol{H} + \eta\nabla^2\boldsymbol{v}.$$

Using equation (7.6) which gives an alternative form of the electromagnetic force $\frac{\mu}{c}\boldsymbol{j}\times\boldsymbol{H}$ we finally arrive at the equation

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\text{grad}\left(p + \frac{\mu H^2}{8\pi}\right) + \frac{\mu}{4\pi}(\boldsymbol{H}.\text{grad})\boldsymbol{H} + \eta\nabla^2\boldsymbol{v}. \tag{16.3}$$

When the electrical conductivity is finite but variable the equation for the magnetic field is again obtained by eliminating $\boldsymbol{E}$ between equations (2.2) and (3.10). However, allowance must now be made for the fact that η_m is a variable, and so this time we proceed by first combining equations (2.9) and (3.10) to give,

$$\frac{c}{4\pi\sigma}\,\text{curl}\,\boldsymbol{H} = \boldsymbol{E} + \frac{\mu}{c}(\boldsymbol{v}\times\boldsymbol{H})$$

from which, by taking the curl of this result and using equation (2.2), it follows that

$$\frac{\partial \boldsymbol{H}}{\partial t} = \text{curl}\,(\boldsymbol{v}\times\boldsymbol{H}) - \text{curl}\,(\eta_m\,\text{curl}\,\boldsymbol{H}). \tag{16.4}$$

Expanding the first term on the right-hand side and using equations (2.3) and (16.1) then gives the desired result

$$\frac{\partial \boldsymbol{H}}{\partial t} = (\boldsymbol{H}\,.\,\mathrm{grad})\boldsymbol{v} - (\boldsymbol{v}\,.\,\mathrm{grad})\boldsymbol{H} - \mathrm{curl}\,(\eta_m\,\mathrm{curl}\,\boldsymbol{H}). \quad (16.5)$$

The energy equation (8.2) is only significant in incompressible flow when the effect of a temperature distribution $T(\boldsymbol{x}, t)$ is to be investigated. When the temperature effect is unimportant the equation may be disregarded. For the moment we shall not concern ourselves with this problem.

The equations to be considered in the incompressible magnetohydrodynamic flow of a viscous inhomogeneously electrically-conducting fluid are then as follows:

Condition for incompressibility:

$$\mathrm{div}\,\boldsymbol{v} = 0, \quad (\boldsymbol{v}.\mathrm{grad})\rho = 0. \qquad (16.1,\,2)$$

Equation of motion:

$$\rho\frac{D\boldsymbol{v}}{Dt} = -\mathrm{grad}\left(p + \frac{\mu H^2}{8\pi}\right) + \frac{\mu}{4\pi}(\boldsymbol{H}\,.\,\mathrm{grad})\boldsymbol{H} + \eta\nabla^2\boldsymbol{v}. \quad (16.3)$$

Equations for the magnetic field:

$$\frac{\partial \boldsymbol{H}}{\partial t} = (\boldsymbol{H}\,.\,\mathrm{grad})\boldsymbol{v} - (\boldsymbol{v}\,.\,\mathrm{grad})\boldsymbol{H} - \mathrm{curl}\,(\eta_m\,\mathrm{curl}\,\boldsymbol{H}), \quad (16.5)$$

$$\mathrm{div}\,\boldsymbol{H} = 0. \qquad (2.3)$$

§ **17. Parallel steady flow.** We now consider special flows in which $\boldsymbol{H}$ and $\boldsymbol{v}$ are everywhere parallel, which we shall call **parallel flows** and set

$$\boldsymbol{H} = \lambda\boldsymbol{v}. \qquad (17.1)$$

By considering the steady state form of the equations of § 16 above describing viscous incompressible flow we shall now show how these equations may be reduced to a set of equations appropriate to ordinary hydrodynamics without a magnetic field when the density and pressure are suitably re-defined. In order that this property may be demonstrated we first notice that it follows directly from equations (2.3), (16.1) and (17.1) that

$$(\boldsymbol{v}\,.\,\mathrm{grad})\lambda = 0. \tag{17.2}$$

Because the motion is steady, equation (17.2) may be interpreted as a mathematical representation of the fact that λ is constant along a streamline, although it may, like the density, have different values on different streamlines.

Introducing equation (17.1) into equation (16.5), which determines the magnetic field, then gives

$$\mathrm{curl}\,(\eta_m\,\mathrm{curl}\,(\lambda\boldsymbol{v})) = \mathbf{0}, \tag{17.3}$$

while the equation of motion (16.3) becomes

$$\rho(\boldsymbol{v}\,.\,\mathrm{grad})\boldsymbol{v} = -\mathrm{grad}\left(p+\frac{\mu\lambda^2\boldsymbol{v}^2}{8\pi}\right)+\frac{\lambda\mu}{4\pi}(\boldsymbol{v}\,.\,\mathrm{grad})(\lambda\boldsymbol{v})+\eta\nabla^2\boldsymbol{v}.$$

Expanding the second term on the right-hand side and using equation (17.2) this finally becomes

$$\left(\rho-\frac{\lambda^2\mu}{4\pi}\right)(\boldsymbol{v}\,.\,\mathrm{grad})\boldsymbol{v} = -\mathrm{grad}\left(p+\frac{\mu\lambda^2\boldsymbol{v}^2}{8\pi}\right)+\eta\nabla^2\boldsymbol{v}. \tag{17.4}$$

If, now, we define a new density $\tilde{\rho}$ and a new pressure $\tilde{p}$ by the expressions

$$\tilde{\rho} = \rho-\frac{\lambda^2\mu}{4\pi} \tag{17.5}$$

and

$$\tilde{p} = p+\frac{\mu\lambda^2\boldsymbol{v}^2}{8\pi} \tag{17.6}$$

then, for parallel steady flow, equations (16.1) and (17.4) become

$$\mathrm{div}\,\boldsymbol{v} = 0 \tag{16.1}$$

and

$$\tilde{\rho}(\boldsymbol{v}\,.\,\mathrm{grad})\boldsymbol{v} = -\mathrm{grad}\,\tilde{p}+\eta\nabla^2\boldsymbol{v}. \tag{17.7}$$

Equation (17.7) is exactly the steady state form of the **Navier-Stokes** equation† for an incompressible fluid flowing with velocity $\boldsymbol{v}$ and having density $\tilde{\rho}$ and pressure $\tilde{p}$. It follows directly from equations (16.2), (17.2) and (17.5) that

$$(\boldsymbol{v}.\text{grad})\tilde{\rho} = 0, \tag{17.8}$$

showing that $\tilde{\rho}$ is constant along the streamlines.

So far the only requirement that need be satisfied by λ is that it must be a constant along each streamline, otherwise it is arbitrary. When the value of this constant is specified on each streamline and suitable boundary conditions are imposed equations (17.3), (16.1) and (17.1) must then be solved in order to determine the incompressible parallel steady flow of an electrically-conducting and viscous fluid in terms of the new fluid density $\tilde{\rho}$ and pressure $\tilde{p}$. The value of λ that is assigned to each streamline together with the value of $\boldsymbol{v}$ determined by the problem can then be used with equation (17.1) to obtain $\boldsymbol{H}$.

We shall now examine two obvious cases in which these equations simplify considerably and permit a direct analogy with ordinary hydrodynamics. Since in ordinary hydrodynamics there is no analogue to equations (16.5) or (17.3) determining the magnetic field, it follows at once that if we are to succeed in our attempt to find reducible problems in magnetohydrodynamic parallel steady flow we must seek those solutions for which equation (17.3) vanishes identically.

Two possible cases occur. In one of these the fluid is perfectly conducting and so $\eta_m = c^2/4\pi\mu\sigma$ vanishes, reducing the problem to the solution of the classical Navier-Stokes equation for incompressible flow. In the second, η_m and λ are constant and

$$\boldsymbol{v} = \text{grad}\,\phi. \tag{17.9}$$

When $\boldsymbol{v}$ is expressible in this form in terms of a scalar

† Compare this result with equation (7.5).

potential function ϕ the flow is called **potential flow.**† In this case equations (16.1) and (17.9) show that ϕ is harmonic since it must satisfy Laplace's equation

$$\nabla^2\phi = 0. \tag{17.10}$$

Consequently a solution may be obtained by solving equation (17.10) for the scalar potential ϕ, taking into account the specified boundary conditions of the problem, and then determining $\boldsymbol{v}$ and $\boldsymbol{H}$ from equations (17.9) and (17.1), respectively. When this has been done the pressure may be found from equations (17.6) and (17.7). In the case that λ and η_m are homogeneous, we have thus demonstrated that any arbitrary potential flow of an ordinary incompressible fluid in the absence of a magnetic field provides a solution to the magnetohydrodynamic parallel steady flow equations (17.3), (16.1) and (17.7).

The parallel steady flow of a perfectly conducting fluid simplifies still further if the fluid is inviscid, for then the only equations determining the motion are the incompressibility conditions

$$\operatorname{div} \boldsymbol{v} = 0, \quad (\boldsymbol{v}.\operatorname{grad})\tilde{\rho} = 0 \tag{17.11}$$

and the equation of motion

$$\tilde{\rho}(\boldsymbol{v}.\operatorname{grad})\boldsymbol{v} = -\operatorname{grad} \tilde{p}. \tag{17.12}$$

These equations in fact describe the ordinary incompressible flow of a conventional fluid in terms of the variables $\boldsymbol{v}$, $\tilde{\rho}$ and $\tilde{p}$. Indeed, by considering any classical incompressible inviscid flow and identifying the velocity, density and pressure with $\boldsymbol{v}$, $\tilde{\rho}$ and $\tilde{p}$, respectively, we can derive a possible incompressible magnetohydrodynamic flow in the variables v, ρ and p. We have already seen from equations (16.2) and (17.2) that although ρ and λ are constant along a streamline they may be inhomogeneous in

† See Rutherford, *Fluid Dynamics*, 1959, p. 5 and Chapter 2, and also Rutherford, *Vector Methods*, 1954, Chapter 7.

the sense that they vary from streamline to streamline. However, since it follows from equation (17.7) that $\tilde{\rho}$ is also constant along a streamline, ρ and λ cannot both be assigned arbitrary values independently of one another and they are in fact related by equation (17.5).

Any inhomogeneity in λ has the interesting consequence that a mapping from a classical hydrodynamical potential flow, for which $\tilde{\rho}$ is everywhere constant and curl $\boldsymbol{v} = \mathbf{0}$, to a magnetohydrodynamic flow results in a flow which will, in general, be inhomogeneous in ρ and such that curl $\boldsymbol{H} \neq \mathbf{0}$. This means that a simple classical fluid flow will generally map into an interesting magnetohydrodynamic flow whenever λ is inhomogeneous. To see this we need only refer to equation (17.5) and expand curl $\boldsymbol{H}$ using equation (17.1) to obtain

$$\text{curl}\,\boldsymbol{H} = \lambda\,\text{curl}\,\boldsymbol{v} + (\text{grad}\,\lambda)\times\boldsymbol{v} = (\text{grad}\,\lambda)\times\boldsymbol{v} \neq \mathbf{0}.$$

A further connection with classical fluid dynamics can be obtained by using the vector identity

$$(\boldsymbol{v}.\text{grad})\boldsymbol{v} = \tfrac{1}{2}\,\text{grad}\,\boldsymbol{v}^2 - \boldsymbol{v}\times\text{curl}\,\boldsymbol{v}$$

to rewrite equation (17.12) in the form

$$\tilde{\rho}\,\text{grad}\,\tfrac{1}{2}\boldsymbol{v}^2 = \boldsymbol{v}\times\text{curl}\,\boldsymbol{v} - \text{grad}\,\tilde{p}. \qquad (17.12')$$

Taking the scalar product of this equation with $\boldsymbol{v}$ and using the properties of a triple scalar product then gives

$$\tilde{\rho}(\boldsymbol{v}.\text{grad})\tfrac{1}{2}\boldsymbol{v}^2 = (\boldsymbol{v}.\text{grad})\tilde{p},$$

or, using the second equation of (17.11),

$$(\boldsymbol{v}.\text{grad})(\tfrac{1}{2}\tilde{\rho}\boldsymbol{v}^2 + \tilde{p}) = 0. \qquad (17.13)$$

Consequently, along each streamline, we must have the relation

$$\tfrac{1}{2}\tilde{\rho}\boldsymbol{v}^2 + \tilde{p} = \text{constant}. \qquad (17.14)$$

Referring to equations (17.5) and (17.6) shows that equation (17.14) may be written as

$$\tfrac{1}{2}\tilde{\rho}\boldsymbol{v}^2 + \tilde{p} = \tfrac{1}{2}\rho\boldsymbol{v}^2 + p = \text{which is constant along a streamline,} \qquad (17.15)$$

and in this form the result will be referred to as **Bernoulli's equation for incompressible parallel flow.**

When the fluid is perfectly conducting but the viscosity cannot be neglected we are required to integrate equation (17.7) subject to the incompressibility conditions (17.11). This is a familiar problem of classical hydrodynamical flow but a slight complication arises when we endeavour to interpret it in terms of incompressible magnetohydrodynamic flow. The difficulty arises from the fact that the fluid density $\tilde{\rho}$ of the classical flow is related to λ and to the density ρ of the magnetohydrodynamic flow by equation (17.5), and so can become negative for some values of ρ and $\boldsymbol{v}$. We now show how this trouble can be resolved when λ is homogeneous everywhere in the flow field.

We first write $\tilde{\rho}$ in the form

$$\tilde{\rho} = \rho\left(1 - \frac{\lambda^2\mu}{4\pi\rho}\right), \tag{17.16}$$

from which it is clear that $\tilde{\rho} \gtrless 0$ according as $1 - \dfrac{\lambda^2\mu}{4\pi\rho} \gtrless 0$, and that $\tilde{\rho} = 0$ when $\dfrac{\lambda^2\mu}{4\pi\rho} = 1$. When $\tilde{\rho} = 0$, it follows from equation (17.1) that the magnitude v of the velocity $\boldsymbol{v}$ is

$$v = \sqrt{\frac{\mu \boldsymbol{H}^2}{4\pi\rho}}. \tag{17.17}$$

For inviscid fluids v becomes the local Alfvén velocity already defined in conjunction with equation (11.3). Equation (16.1) again serves to determine the velocity while equation (17.6) determines the pressure for, when $\tilde{\rho} = \eta = 0$, it follows from equation (17.7) that $\tilde{p}$ = constant.

In the general viscous case $\tilde{\rho} \gtrless 0$ according as v is greater or less than the local Alfvén velocity; that is according as

$v \gtrless \sqrt{\dfrac{\mu \boldsymbol{H}^2}{4\pi\rho}}$. Both of these cases may easily be reduced to the solution of a classical steady viscous flow problem in the absence of a magnetic field by the following simple change of variables. We define a density ρ^+, a pressure p^+ and a velocity $\boldsymbol{v}^+$ by the relations

$$\rho^+ = \varepsilon\tilde{\rho}, \quad p^+ = \varepsilon\tilde{p}+\text{constant}, \quad \boldsymbol{v}^+ = \varepsilon\boldsymbol{v}, \qquad (17.18)$$

where ε is the sign of the expression $v - \sqrt{\dfrac{\mu \boldsymbol{H}^2}{4\pi\rho}}$. It is then easily seen that the density ρ^+ is always positive, as is the pressure if the constant is suitably chosen. The vector $\boldsymbol{v}^+$ still satisfies the incompressibility condition

$$\operatorname{div} \boldsymbol{v}^+ = 0, \qquad (17.19)$$

while ρ^+, p^+ and $\boldsymbol{v}^+$ satisfy the classical steady state form of the Navier-Stokes equation for viscous incompressible flow

$$\rho^+(\boldsymbol{v}.\text{grad})\boldsymbol{v}^+ = -\text{grad}\, p^+ + \eta\nabla^2\boldsymbol{v}^+, \qquad (17.20)$$

which we have already encountered in equation (17.7). Thus, once again, the problem of steady parallel magnetohydrodynamic flow has been related to a classical steady state hydrodynamical flow.

§ **18. One-dimensional steady viscous flow.** Other simple magnetohydrodynamic steady flows arise from flow configurations in which either one or two of the coordinate variables involved are ignorable.† These flows have considerable practical significance since they describe the steady flow of fluids in ducts and pipes of constant cross-sectional area where the flow is induced either by a pressure

† We use the term **ignorable** here in the sense that although we are describing a real flow in three space dimensions, one or more of the coordinate variables is absent from the mathematical formulation of the problem.

gradient along the duct or by the motion of the duct walls relative to one another. We shall not attempt a discussion of the general problem, being content instead to study the class of flows that occur between two infinite parallel planes, for which all flow quantities at any point in the fluid depend only on the perpendicular distance of that point from one of the bounding planes. Two special examples of this type of flow known as Hartmann flow and Couette flow will be examined in rather more detail since they have many useful applications.

When entropy is constant along a streamline the fact can be expressed mathematically by the relation

$$\frac{DS}{Dt} = 0 \tag{18.1}$$

which, in the case of steady flow, becomes

$$(\boldsymbol{v}\,.\,\mathrm{grad})S = 0. \tag{18.2}$$

Flows for which this condition is satisfied are called **isentropic flows** and it should be noticed that the constant value of the entropy may differ for different fluid elements on different streamlines. When the entropy has the same constant value throughout the entire fluid the flow is said to be **homentropic**. For the rectilinear flows which we shall now study, the left-hand side of the energy equation (8.2) vanishes and the resulting equation then serves to determine the temperature distribution when the flow velocity and magnetic field have been found.

In the case of rectilinear steady flow between two parallel planes it is convenient to select a right-handed set of Cartesian axes $O\{x, y, z\}$ so that the z-axis lies parallel to the flow velocity v_z which, intuitively, we would expect to be parallel to the plane walls of the duct. The x-axis will be taken perpendicular to the duct walls. Thus we shall consider steady flows in which only the z-component $v_z = v_z(x)$ of the velocity vector $\boldsymbol{v}$ is non-zero and where the magnetic induction vector $\boldsymbol{B} = \boldsymbol{B}(x)$ is constant along streamlines. Of the equations describing the fluid motion, equation (18.2) is satisfied automatically, whilst for a fluid of constant density the continuity equation reduces to the incompressibility condition $\operatorname{div} \boldsymbol{v} = 0$ given in equation (16.1) and is satisfied identically in the flow under consideration. For a viscous fluid in which the effects of external forces may be neglected, the form of the equation of motion with which we shall choose to work is the steady state form of the equation displayed in (16.3). The current $\boldsymbol{j}$ can be determined from equation (2.9) when $\boldsymbol{H}$ is known.

Because the flow is assumed to be steady, accelerations do not occur and hence the fluid density is immaterial. Differentiation of equation (7.5) with respect to z yielding $\operatorname{grad}(\partial p/\partial z) = 0$ shows that $\partial p/\partial z$ is a constant which we shall denote by $-k_1$, a result we would expect since the solution is unchanged by translations along the z-axis.

The x, y and z-components of the equation of motion (16.3) then become, respectively,

$$0 = -\frac{\partial}{\partial x}\left(p + \frac{\mu \boldsymbol{H}^2}{8\pi}\right) + \frac{\mu}{4\pi} H_x \frac{\partial H_x}{\partial x}, \tag{18.3}$$

$$0 = \frac{\mu}{4\pi} H_x \frac{\partial H_y}{\partial x}, \tag{18.4}$$

$$0 = k_1 + \frac{\mu}{4\pi} H_x \frac{\partial H_z}{\partial x} + \eta \frac{\partial^2 v_z}{\partial x^2}. \tag{18.5}$$

The magnetic field is governed by equations (2.3) and (16.5) which, when a fluid having constant electrical conductivity flows in the duct, can be expressed in component notation as

$$\frac{\partial H_x}{\partial x} = 0, \tag{18.6}$$

and

$$\frac{\partial^2 H_x}{\partial x^2} = 0, \tag{18.7}$$

$$\frac{\partial^2 H_y}{\partial x^2} = 0, \tag{18.8}$$

$$\eta_m \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial}{\partial x}(v_z H_x) = 0, \tag{18.9}$$

repectively. These equations simplify considerably on inspection since it is at once apparent from equations (18.3) and (18.6) that

$$p + \frac{\mu \boldsymbol{H}^2}{8\pi} = F(y, z),$$

where, for the moment, $F(y, z)$ is an arbitrary function of y and z. However, since by hypothesis p and $\boldsymbol{H}$ are independent of y, and $\frac{\partial \boldsymbol{H}}{\partial z} = 0$ and $\frac{\partial p}{\partial z} = -k_1$, it at once follows that the most general form of F is $F = k_2 - k_1 z$, where k_2 is a constant. Thus the pressure p and the magnetic field $\boldsymbol{H}$ are related by the equation

$$p + \frac{\mu \boldsymbol{H}^2}{8\pi} = k_2 - k_1 z. \tag{18.10}$$

Now it follows from equation (18.6) that H_x = constant $= H_0$, say, which result, when used in equation (18.4), shows that H_y = constant $= H_1$, say. This shows that equations (18.7) and (18.8) are satisfied automatically and so are redundant from the point of view of this analysis.

The original problem has now been reduced to the solution of the simultaneous, linear, constant coefficient ($H_x = H_0$), partial differential equations (18.5) and (18.9) for $v_z(x)$ and $H_z(x)$. A specific flow will of course only be determined when the boundary conditions appropriate to a definite problem have been imposed. Equation (18.10) can then be used in conjunction with the solutions for v_z and H_z in order to determine the pressure p.

The fact that we have succeeded in describing the problem in terms of our initial heuristic assumptions regarding the flow, without encountering incompatibilities amongst the resulting equations, establishes the correctness of these assumptions. Indeed, the compatibility of the resulting equations is in fact closely related to the redundancy of equations (18.7) and (18.8). This is so because we were able to find solutions common to equations (18.4) and (18.8) and to equations (18.6) and (18.7). If, for example, H_y was linear in x it would still be a solution of equation (18.8) but, unless $H_0 \equiv 0$, it would no longer satisfy equation (18.4) and the flow would cease to be rectilinear, and thus capable of description only in terms of x.

In a sense z is not strictly an ignorable coordinate since it appears in equation (18.10) determining the pressure, but we shall regard it as such since p is determined automatically when v_z and H_z have been found from equations (18.5) and (18.9). With this in mind we see that the governing equations (18.5) and (18.9) may be written as the ordinary differential equations

$$\eta \frac{d^2 v_z}{dx^2} + \frac{\mu H_0}{4\pi} \frac{dH_z}{dx} + k_1 = 0 \tag{18.11}$$

and

$$\eta_m \frac{d^2 H_z}{dx^2} + H_0 \frac{dv_z}{dx} = 0. \tag{18.12}$$

There are many methods by which these equations may be solved † but we shall proceed as follows. Multiply equation (18.12) by a constant α and add it to equation

† See, for example, Ince, *Integration of Ordinary Differential Equations*, 1952, § 51.

(18.11) to obtain

$$\eta \frac{d^2}{dx^2}\left(v_z + \frac{\alpha\eta_m}{\eta} H_z\right) + \alpha H_0 \frac{d}{dx}\left(v_z + \frac{\mu H_z}{4\pi\alpha}\right) + k_1 = 0. \quad (18.13)$$

Then, if we set

$$u = v_z + \frac{\alpha\eta_m}{\eta} H_z, \quad (18.14)$$

and α is chosen so that

$$\alpha = \left(\frac{\mu\eta}{4\pi\eta_m}\right)^{\frac{1}{2}}, \quad (18.15)$$

equation (18.13) becomes

$$\eta \frac{d^2u}{dx^2} + \alpha H_0 \frac{du}{dx} + k_1 = 0. \quad (18.16)$$

Similarly, multiplying equation (18.12) by α and subtracting it from equation (18.11) enables us to define w by the relation

$$w = v_z - \frac{\alpha\eta_m}{\eta} H_z, \quad (18.17)$$

where α is again given by equation (18.15). The resulting equation then becomes

$$\eta \frac{d^2w}{dx^2} - \alpha H_0 \frac{dw}{dx} + k_1 = 0. \quad (18.18)$$

Since $\eta_m = c^2/4\pi\mu\sigma$, equations (18.16) and (18.18) determining the flow in terms of u and w become

$$\eta \frac{d^2u}{dx^2} + (\sigma\eta/c^2)^{\frac{1}{2}} B_0 \frac{du}{dx} + k_1 = 0 \quad (18.16')$$

and

$$\eta \frac{d^2w}{dx^2} - (\sigma\eta/c^2)^{\frac{1}{2}} B_0 \frac{dw}{dx} + k_1 = 0, \quad (18.18')$$

where

$$u = v_z + \frac{c}{4\pi\mu}(\sigma\eta)^{-\frac{1}{2}}B_z \tag{18.14'}$$

and

$$w = v_z - \frac{c}{4\pi\mu}(\sigma\eta)^{-\frac{1}{2}}B_z. \tag{18.17'}$$

§ 19. **Hartmann flow.** We are now in a position to apply the results of the previous section to an actual flow problem and we shall start by considering the case of **Hartmann flow.** In this steady flow of a viscous fluid, a magnetic field of strength B_0 is imposed normal to the surface of non-conducting fixed duct walls which are assumed to be a distance $2d$ apart and flow is induced by a pressure gradient along the duct. The origin of the coordinates will be assumed to be located at the centre of the duct (see Fig. 7).

The magnetic induction vector $\boldsymbol{B}_0$ must then satisfy the boundary condition (15.8) across each of the infinite parallel planes comprising the duct walls, from which it follows at once that $B_z(-d) = B_z(d) = 0$. Since the fluid is viscous, the fluid kinematic boundary conditions on the walls are obviously that $v_z(-d) = v_z(d) = 0$.

Equations (18.14′) and (18.17′) then show that the boundary conditions for u and w on the walls of the duct are $u(-d) = u(d) = 0$ and $w(-d) = w(d) = 0$. Having now determined the boundary conditions we may seek the appropriate solution of equations (18.16′) and (18.18′).

Considering equation (18.16′) first, we see that it is a linear constant coefficient second order differential equation. Thus we may expect a solution comprising terms of the form $u = e^{mx}$. Substituting this expression into equation (18.16′) then leads to the auxiliary equation †

$$\eta m^2 + (\sigma\eta/c^2)^{\frac{1}{2}}B_0 m = 0, \tag{19.1}$$

with roots

$$m = 0, \quad m = -(\sigma/c^2\eta)^{\frac{1}{2}}B_0. \tag{19.2}$$

† Ince, *loc. cit.*, § 39.

Defining the non-dimensional number

$$R_h = (\sigma/c^2\eta)^{\frac{1}{2}} B_0 d, \tag{19.3}$$

which is called the **Hartmann number,** we see that the non-zero root of the auxiliary equation defines a term $\exp(-R_h x/d)$ in the general solution for u. The zero root

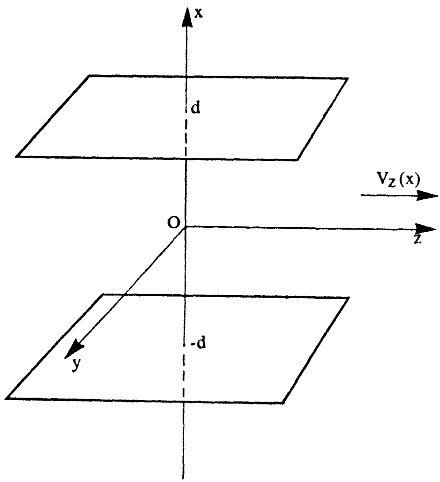

FIG. 7.

corresponds to a term in the general solution which is linearly dependent on x, and a simple calculation then shows that the general solution for u is

$$u = A \exp(-R_h x/d) - \frac{k_1 x d}{\eta R_h} + B. \tag{19.4}$$

A similar argument applied to equation (18.18′) shows that the general solution for w is

$$w = C \exp(R_h x/d) + \frac{k_1 x d}{\eta R_h} + D, \tag{19.5}$$

where A, B, C and D are arbitrary constants.

Equations (19.4) and (19.5) comprise the general solution to equations (18.16′) and (18.18′) and the solution to the Hartmann flow problem follows directly when the arbitrary constants are chosen in accordance with the boundary conditions on u and w. Notice that the pressure gradient along the z-axis, $\dfrac{\partial p}{\partial z} = -k_1$, is itself a condition which must be specified, as must the reference level of pressure k_2.

Applying the boundary conditions to these general solutions and using equations (18.14′) and (18.17′) then gives the desired results

$$v_z(x) = \left(\frac{k_1 d^2}{\eta R_h}\right)\left\{\frac{\cosh R_h - \cosh (R_h x/d)}{\sinh R_h}\right\} \tag{19.6}$$

and

$$B_z(x) = \frac{4\pi\mu}{c}(\sigma\eta)^{\frac{1}{2}}\left(\frac{k_1 d^2}{\eta R_h}\right)\left\{\frac{\sinh (R_h x/d) - (x/d)\sinh R_h}{\sinh R_h}\right\}. \tag{19.7}$$

An alternative expression of the solution in terms of the flow velocity v_0 at the mid-plane can be obtained by using equation (19.6) and the definition $v_0 = v_z(0)$ to obtain

$$v_0 = \left(\frac{k_1 d^2}{\eta R_h}\right)\left\{\frac{\cosh R_h - 1}{\sinh R_h}\right\}. \tag{19.8}$$

This expression then relates the pressure gradient $\dfrac{\partial p}{\partial z} = -k_1$ and v_0, which has the same sign as k_1, and enables us to express v_z and H_z in the alternative form

$$v_z(x) = v_0\left\{\frac{\cosh R_h - \cosh (R_h x/d)}{\cosh R_h - 1}\right\} \tag{19.9}$$

and

$$B_z(x) = v_0\frac{4\pi\mu}{c}(\sigma\eta)^{\frac{1}{2}}\left\{\frac{\sinh (R_h x/d) - (x/d)\sinh R_h}{\cosh R_h - 1}\right\}. \tag{19.10}$$

The pressure p is determined by equation (18.10) which now simplifies to

$$p+\frac{\mu}{8\pi}(H_0^2+H_z^2)=k_2-k_1 z, \tag{19.11}$$

with the constant k_2 determining the reference level of pressure.

Two limiting cases are of interest and occur when the magnetic field becomes vanishingly small ($R_h \to 0$) and when it becomes very large ($R_h \gg 1$). In the first case the flow tends in the limit to the **Poiseuille flow** of an ordinary viscous fluid between parallel fixed plane boundaries, and the velocity $v_z(x)$ of equation (19.9) becomes †

$$v_z(x)=v_0\left(1-\frac{x^2}{d^2}\right). \tag{19.12}$$

In the second case the velocity v_z becomes asymptotic to

$$v_z(x) = v_0\{1-\exp[-R_h(1-|x|/d)]\}, \tag{19.13}$$

which shows that for strong magnetic fields the velocity profile in the duct becomes almost flat over most of the channel with the transition to a zero velocity at the boundaries being virtually confined to a thin boundary layer adjacent to each wall.

The effect of R_h on the velocity profile v_z, expressed as a function of x with the pressure gradient kept constant, is shown in Fig. 8 for representative values of R_h.

Since the gradient $\dfrac{dz}{dx}$ of a magnetic line of force in the (x, z)-plane is determined by $\dfrac{dz}{dx}=\dfrac{B_z}{B_0}$, we can use equation (19.10) to find their shape. The interpretation of our results will become much more general if we first re-write equation (19.10) in terms of the magnetic Reynolds number $\mathscr{R}_m$

† Cf. Rutherford, *Fluid Dynamics*, 1959, § 60.

introduced in expression (10.10). To do this we identify the characteristic length L with d and the characteristic velocity V with v_0 when

$$\mathscr{R}_m = 4\pi\mu\sigma v_0 d/c^2. \tag{19.14}$$

Written in terms of $\mathscr{R}_m$ equation (19.10) becomes

$$\frac{B_z}{B_0} = \frac{\mathscr{R}_m}{R_h}\left\{\frac{\sinh(R_h x/d)-(x/d)\sinh R_h}{\cosh R_h - 1}\right\}, \tag{19.15}$$

from which the shape of a magnetic line of force is easily seen to be given by the equation

$$z = \frac{\mathscr{R}_m d}{R_h^2}\left\{\frac{\cosh(R_h x/d)-(x^2/2d^2)R_h \sinh R_h}{\cosh R_h - 1}\right\} + \text{const.} \tag{19.16}$$

Equations (19.15) and (19.16) show that the magnetic lines of force leave the duct walls at right angles and then bulge

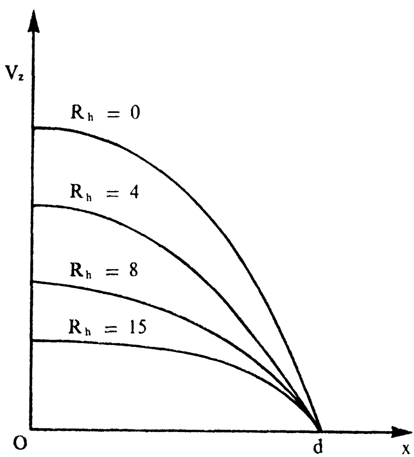

FIG. 8.

in the direction of the fluid flow as the conducting fluid tends to drag them along. The size of the bulge is seen to be directly proportional to $\mathscr{R}_m$ for a fixed Hartmann number R_h.

A representative set of profiles of magnetic lines of force are illustrated in Fig. 9. These curves have been

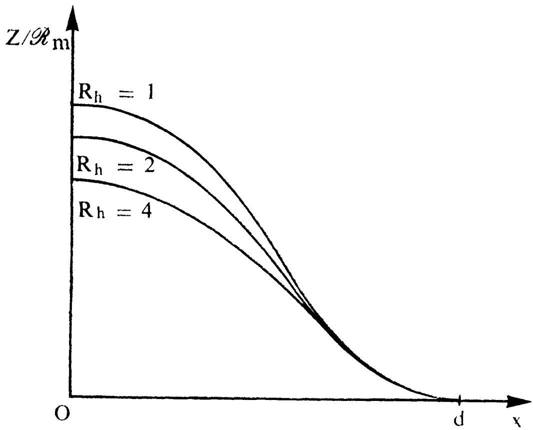

FIG. 9.

obtained by keeping the magnetic Reynolds number $\mathscr{R}_m$ constant and by adjusting the constant of integration in equation (19.16) so as to make all the curves tangent to the x-axis at the points $x = \pm d$.

§ **20. Couette flow.** We have seen that one effect of the transverse magnetic field in Hartmann flow was to flatten the fluid velocity profile. This flattening caused the transition from the free stream flow conditions near the centre to the stationary layer of fluid adjacent to either wall to be confined to thin **boundary layers** adjacent to each wall. Since there are only clearly defined boundary layers when

$R_h \gg 1$ we shall start by making use of equation (19.13) to obtain a rough estimate of their thickness. To do this we need only notice that v_z reduces from its maximum value v_0 at the centre to a value $v_0(1-1/e) \simeq 2v_0/3$ at a distance $|x| = (1-1/R_h)$ from the centre line. This conveniently defines regions of width d/R_h, adjacent to each fixed boundary, in which most of the change in the fluid velocity profile takes place. Since the extent of the boundary layer is somewhat arbitrary it will be convenient for our purposes if we identify the boundary layers with these two regions of thickness d/R_h whenever $R_h \gg 1$. The fluid motion in such a boundary layer is of considerable interest and can be approximated by the steady viscous shear flow that takes place between parallel plane boundaries when the upper boundary moves parallel to itself with a constant velocity V relative to the lower one which is assumed to be fixed; there being no pressure gradient imposed along the direction of flow. In this approximation the moving boundary then represents the streamline in the fluid that forms the edge of the boundary layer. Shear flow of this type is known as **Couette flow** and also provides an approximation to the more general boundary layer flow that takes place over a body of revolution of large radius of curvature when it moves at a constant velocity along its axis. (The flow here being referred to axes fixed in the body.)

In order that we may analyse this flow let us now assume that it takes place between parallel planes a distance d apart and that the coordinate system of § 18 is used with the origin located on the fixed lower plane. The kinematic boundary conditions for the fluid are then $v_z(0) = 0$ and $v_z(d) = V$. The formulation of a consistent set of electromagnetic boundary conditions now becomes more complicated than in the previous problem but, as before, we can still impose a constant x-component of magnetic induction B_0. Since the flow is steady it is an immediate consequence of equation (2.2) that $\boldsymbol{E} = \boldsymbol{E}_0$ is a constant, the value of which is determined by the assumptions we choose to make about the field. The relationship between $\boldsymbol{E}_0$ and $\boldsymbol{H}$ is easily established by combining equations (2.9) and (3.10) to give

$$\operatorname{curl} \boldsymbol{H} = \frac{4\pi\sigma}{c}\left(\boldsymbol{E}_0 + \frac{1}{c}\,\boldsymbol{v}\times\boldsymbol{B}\right). \tag{20.1}$$

By considering the kinematic conditions on the fixed wall this equation provides the following boundary condition on $x = 0$,

$$\frac{\partial H_z}{\partial x} = -\frac{4\pi\sigma}{c}E_{y0} \tag{20.2}$$

where E_{y0} is the y-component of $\boldsymbol{E}_0$. The nature of this lower wall has no effect on the flow. This is so because as the wall is stationary with respect to the magnetic field, even if it were to be a conductor, no currents would be induced in it. However, this is not true for the upper boundary which moves relative to the magnetic field, and so to simplify the problem we shall consider that the moving boundary is an insulator. When this flow is used as an approximation to the boundary layer flow over a body moving in an infinite medium it is reasonable to assume that $\boldsymbol{E} = \boldsymbol{0}$ at infinity and, consequently, that $E_{y0} = 0$ (compare with Example 5, § 22).

Now the equations (18.16′) and (18.18′) governing the flow are both second order equations and so require a total of four boundary conditions to be specified on the two plane boundaries if a solution is to be uniquely determined. Two of these comprise the kinematic boundary conditions whilst a third is provided by equation (20.2). For the fourth condition we may specify the component B_z on either the fixed or the moving boundary. We choose to examine the case $B_z = 0$ on $x = 0$.

The boundary conditions for the problem in terms of

v_z and B_z are thus

$$v_z = 0,\ B_z = 0,\ \frac{\partial B_z}{\partial x} = 0 \text{ on } x = 0 \qquad (20.3a)$$

and

$$v_z = V \text{ on } x = d. \qquad (20.3b)$$

Using equations (18.14′) and (18.17′) to interpret these in terms of boundary conditions on u and w gives

$$u = 0,\ w = 0,\ \frac{\partial u}{\partial x} - \frac{\partial w}{\partial x} = 0 \text{ on } x = 0 \quad (20.4a)$$

and

$$u + w = 2V \text{ on } x = d. \qquad (20.4b)$$

The expressions for v_z and B_z which result when these boundary conditions are used with the general solutions (19.4) and (19.5) to equations (18.16′) and (18.18′), together with the fact that there is no pressure gradient ($k_1 = 0$), are

$$v_z(x) = \frac{V \sinh (R_h x/d)}{\sinh R_h} \qquad (20.5)$$

and

$$B_z = \frac{4\pi\mu(\sigma\eta)^{\frac{1}{2}}V}{c \sinh R_h} \{1 - \cosh (R_h x/d)\}, \qquad (20.6)$$

where again $R_h = (\sigma/c^2\eta)^{\frac{1}{2}} B_0 d$. Velocity profiles for representative values of R_h are shown in Fig. 10. The limiting profile corresponding to $R_h \to 0$ is linear in x as may be easily seen from equation (20.5).

The effect of increasing the Hartmann number R_h is to reduce the flow velocity throughout the entire flow region and to decrease the velocity gradient in the vicinity of the fixed boundary while increasing it at the moving boundary.

It is a physically observed fact in the steady flow of ordinary fluids that when the rate of shear between fluid layers becomes too great the motion ceases to be smooth and an unsteady irregular motion known as **turbulence** takes place. The condition for the onset of turbulence in ordinary fluid flows is expressed in terms of a critical Reynolds number $\mathscr{R}_c$ appropriate to the type of flow involved and which, when exceeded, is associated with turbulent flow. This would indicate that turbulence must also play an important part in determining the extent to

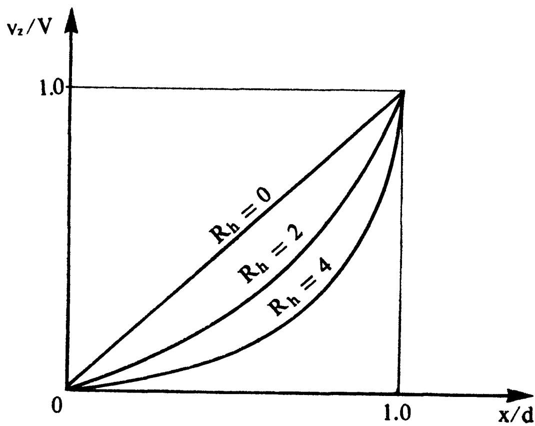

FIG. 10.

which the flow of a conducting fluid may be continuously deformed by a magnetic field before the fluid motion becomes irregular and hydromagnetic turbulence occurs. The analysis of turbulent magnetohydrodynamic motion is very complicated and will not be pursued here. We should however remark in passing that high values of R_h produce both an appreciable retardation of the flow velocity in the boundary layer and very large velocity gradients at the moving boundary which could lead to turbulent boundary layer flow.

§ 21. Temperature distribution. The energy dissipation in the fluid due to viscous effects and to Joule heating causes a non-uniform temperature distribution throughout the fluid flow. This temperature distribution is described by the energy equation (8.2) and, in general flow problems, can only be determined when the energy equation is solved simultaneously with the other equations describing the fluid and electromagnetic properties. However the problem becomes much simpler in the case of the incompressible flows that we are considering, for then the energy equation becomes decoupled from the other equations and can be solved separately when $v_z(x)$ and $H_z(x)$ have been found. This is so because the assumption of incompressibility precludes the possibility of temperature variations influencing the fluid equations (we ignore convection effects).

As was already noted in § 18, since we are concerned with rectilinear steady flows in which all quantities are of necessity constant along streamlines, the left-hand side of the energy equation (8.2) vanishes and the right-hand side reduces to the ordinary linear differential equation

$$0 = \eta \left(\frac{dv_z}{dx} \right)^2 + \frac{c^2}{16\pi^2\sigma} \left(\frac{dH_z}{dx} \right)^2 + \chi \frac{d^2T}{dx^2}, \tag{21.1}$$

in which $v_z(x)$ and $H_z(x)$ are assumed to be known and χ is assumed to be constant.

By using equation (21.1) together with two suitable boundary conditions the temperature distribution $T(x)$ and the heat flow through the boundaries can easily be found. These boundary conditions may take a number of forms, one of the simplest of which is the requirement that a boundary surface be maintained at a given temperature $T = T_0$. Alternatively, since the heat flux q is proportional to the temperature gradient and flows in the direction of decreasing temperature, the heat flux q itself, given by $q = -\chi \dfrac{dT}{dx}$, may be specified on the boundary. A thermally insulated boundary is thus described by the boundary condition $\dfrac{dT}{dx} = 0$. Another boundary condition is that corresponding to **Newton's law of cooling** in which the transport of heat energy across a boundary at a temperature T is proportional to the temperature difference $T - T_0$ between the boundary and the adjacent medium which is assumed to be at a constant temperature T_0. This boundary condition has the mathematical representation

$$h(T - T_0) + \chi \frac{dT}{dx} = 0$$

and is a good approximation provided the temperature difference $T - T_0$ is small. It contains the two previous boundary conditions as the limiting cases $h/\chi \to \infty$ and $h/\chi \to 0$, respectively, and although usually known as the **radiation boundary condition** is better considered as Newton's law of cooling since true radiant heat loss at high temperature is non-linear and varies as T^4.

To illustrate the method of determination of a temperature distribution let us take for our example the Hartmann flow of fluid through a duct described in § 19, and suppose that the duct wall at $x = -d$ is maintained at a temperature T_0 whilst the duct wall at $x = d$ is maintained at a temperature T_1.

Using the values of $v_z(x)$ and $H_z(x)$ determined by equations (19.6) and (19.7) in equation (21.1) gives

$$\frac{d^2T}{dx^2} + \frac{1}{\sinh^2 R_h}\left(\frac{k_1^2 d^2}{\chi\eta}\right)\left\{\cosh(2R_h x/d) - \frac{2\cosh(R_h x/d)\sinh R_h}{R_h} + \frac{\sinh^2 R_h}{R_h^2}\right\} = 0. \qquad (21.2)$$

This simple equation can be integrated directly to obtain the general solution

$$T = -\frac{1}{\sinh^2 R_h}\left(\frac{k_1^2 d^4}{\chi\eta R_h}\right)\left\{\frac{\cosh(2R_h x/d)}{4R_h} - \frac{2\cosh(R_h x/d)\sinh R_h}{R_h^2} + \frac{1}{2}\left(\frac{x}{d}\right)^2 \frac{\sinh^2 R_h}{R_h}\right\} + Ax + B. \tag{21.3}$$

If, now, we apply the boundary conditions to equation (21.3) by requiring that $T = T_0$ when $x = -d$ and $T = T_1$ when $x = d$, and we set

$$K = -\frac{1}{\sinh^2 R_h}\left(\frac{k_1^2 d^4}{\chi\eta R_h}\right)\left\{\frac{\cosh 2R_h}{4R_h} - \frac{2\cosh R_h \sinh R_h}{R_h^2} + \frac{\sinh^2 R_h}{2R_h}\right\}, \tag{21.4}$$

we find that

$$A = (T_1 - T_0)/2d \quad \text{and} \quad B = \tfrac{1}{2}(T_0 + T_1) - K. \tag{21.5}$$

The temperature distribution is thus given by the following expression

$$T = -\frac{1}{\sinh^2 R_h}\left(\frac{k_1^2 d^4}{\chi\eta R_h}\right)\left\{\frac{\cosh(2R_h x/d)}{4R_h} - \frac{2\cosh(R_h x/d)\sinh R_h}{R_h^2} + \frac{1}{2}\left(\frac{x}{d}\right)^2 \frac{\sinh^2 R_h}{R_h}\right\} + \tfrac{1}{2}(T_1 - T_0)(x/d) + \tfrac{1}{2}(T_0 + T_1) - K, \tag{21.6}$$

and the heat flux q through a unit area of any plane $x = x_1$ is given by the expression

$$q = -\chi \left.\frac{dT}{dx}\right|_{x = x_1}. \tag{21.7}$$

Equation (21.7) shows that when $T_0 = T_1$, and so the duct walls are both maintained at the same temperature, the heat flux $q = \chi(T_0 - T_1)/2d$ through the centre plane $x = 0$ vanishes. This result is of course to be expected since the heat flow is then determined only by the dissipative effects in the fluid and, since Hartmann flow is symmetric with respect to the z-axis, there will be no heat flux across the plane of symmetry.

Although we have assumed that the fluid is incompressible, significant changes in fluid density can nevertheless occur as a result of large temperature changes. Consequently, if the flow takes place in a gravitational field and large temperature gradients arise, it can be anticipated that a convection type process will ensue and cause a flow instability in which the flow will cease to be rectilinear, and so our solution will no longer be valid.

Instabilities of this rather simple kind, and of a more complicated type usually associated with the magnetic effects, are of considerable importance since they largely determine which of the *mathematically* possible flow configurations are *physically possible.*

§ 22. Examples. 1.* Defining the enthalpy i per unit mass of a compressible fluid by the relation $i = e + p\tau$, show that $di = TdS + \tau dp$. When the entropy is constant within a region, and so $di = \tau dp$, show the $\operatorname{grad} i = \tau \operatorname{grad} p$ and hence prove that the equation of motion of a classical compressible inviscid fluid in the absence of a magnetic field can be written

$$\frac{\partial \boldsymbol{v}}{\partial t} + \operatorname{grad}(\tfrac{1}{2}\boldsymbol{v}^2 + i) = \boldsymbol{v} \times \operatorname{curl} \boldsymbol{v}.$$

Consequently, when the flow is steady, prove that Bernoulli's equation becomes

$$\tfrac{1}{2}\boldsymbol{v}^2 + i = K,$$

where K is a constant along a streamline and is, in general, different for different streamlines. Show that when curl $\boldsymbol{v} = \mathbf{0}$, and so the flow is **irrotational**, that the introduction of the **velocity potential** ϕ such that $\boldsymbol{v} = \text{grad}\,\phi$ leads to the following form of Bernoulli's equation

$$\frac{\partial \phi}{\partial t} + \tfrac{1}{2}v^2 + i = f(t),$$

where $f(t)$ is a function only of the time t. Hence show that for steady homogeneous incompressible flow Bernoulli's equation becomes

$$\tfrac{1}{2}\rho v^2 + p = K_1,$$

where K_1 is a constant throughout the entire fluid.

2. Consider a **transverse flow** which is defined to be a two-dimensional magnetohydrodynamic flow in the (x, y)-plane such that the magnetic vector $\boldsymbol{B}$ is perpendicular to the plane of flow and, like all other quantities, is independent of z being a function only of x, y and the time t. Denoting the magnitude of $\boldsymbol{B}$ by B and using the Lundquist equations show that the magnetic field equation becomes

$$\frac{\partial B}{\partial t} + \text{div}\,(B\boldsymbol{v}) = 0,$$

or, by the continuity equation,†

$$\frac{D}{Dt}\left(\frac{B}{\rho}\right) = 0;$$

and that the momentum equation becomes

$$\rho \frac{D\boldsymbol{v}}{Dt} + \text{grad}\left(p + \frac{\mu H^2}{8\pi}\right) = \mathbf{0}.$$

Hence show that for flow within a region of constant

† Compare with Example 9, § 12.

entropy Bernoulli's equation takes the form

$$\tfrac{1}{2}v^2+i+\frac{\mu H^2}{8\pi} = \text{which is constant along a streamline,}$$

where $i = e+p\tau$ is the enthalpy per unit mass of the fluid.

3. Consider the Lundquist equations appropriate to the two-dimensional steady incompressible flow of a fluid around a perfectly conducting rigid boundary over some part of which $B_n \neq 0$. Then, by using the electric boundary condition in the fluid and the fact that $\mathbf{E}_t = \mathbf{0}$ on the conductor, show that the fluid velocity $\mathbf{v}$ is identically zero on those parts of the boundary where $B_n \neq 0$. Use the equation for the magnetic field together with Ohm's law for a perfectly conducting fluid to prove that $\mathbf{E}$ is a constant vector perpendicular to the plane of the flow. Hence, by showing that the flow is parallel flow and by considering the solution in the vicinity of the boundary, prove that $\mathbf{v} \equiv \mathbf{0}$ along every magnetic line of force which intersects the boundary.

4. Show that the average fluid velocity $\bar{v}$ in Hartmann flow between parallel planes separated by a distance $2d$ is

$$\bar{v} = \frac{k_1 d^2}{\eta R_h^2}(R_h \coth R_h - 1).$$

Deduce that when $R_h \ll 1$, the average fluid velocity $\bar{v} = k_1 d^2/3\eta$ and that when $R_h \gg 1$, it becomes $\bar{v} = k_1 d^2/\eta R_h$. Show that the components v_z and B_z of the velocity and magnetic induction along the duct may then be written in the form

$$v_z = \bar{v}\left\{\frac{\cosh R_h - \cosh (R_h x/d)}{\cosh R_h - (\sinh R_h)/R_h}\right\}$$

and

$$B_z = \frac{4\pi\mu(\sigma\eta)^{\frac{1}{2}}\bar{v}}{c}\left\{\frac{\sinh (R_h x/d) - (x/d)\sinh R_h}{\cosh R_h - (\sinh R_h)/R_h}\right\}.$$

Then prove that the y-component j_y of the current vector $\boldsymbol{j}$ is given by

$$j_y = -\frac{(\sigma\eta)^{\frac{1}{2}}\bar{v}}{d}\left\{\frac{R_h \cosh(R_h x/d) - \sinh R_h}{\cosh R_h - (\sinh R_h)/R_h}\right\},$$

and that the y-component E_y of the electric field vector $\boldsymbol{E}$ is

$$E_y = -\bar{v}B_0/c = \text{constant}.$$

5. Consider the Couette flow of a conducting fluid between insulating planes a distance d apart in which the magnetic field of strength B_0 is fixed to the moving plane and where no external electric field is present. Use a reference frame with its origin attached to the fixed insulating plane $x = 0$ and assume that the plane $x = d$ moves parallel to itself with velocity V in the z-direction and that the component of the magnetic field H_z parallel to the moving plane is zero at its surface. (The lines of force are normal to the surface of the moving plane.) Show, by considering the constant transverse electric field that is seen relative to the reference frame attached to the fixed plane, that the boundary condition for H_z on $x = 0$ is

$$\partial H_z/\partial x = 4\pi\sigma V B_0/c.$$

Then, using the kinematic boundary conditions $v_z(0) = 0$ and $v_z(d) = V$, show that

$$v_z(x) = V\left\{1 - \frac{\sinh R_h(1-x/d)}{\sinh R_h}\right\}$$

and

$$H_z(x) = \frac{4\pi(\sigma\eta)^{\frac{1}{2}}V}{c \sinh R_h}\{1 - \cosh R_h(1-x/d)\}.$$

6. Show that the temperature distribution in plane Couette flow with a magnetic field of strength B_0 directed normal to a thermally insulated wall relative to which the magnetic field is stationary is

$$T = \frac{1}{\chi}\left\{T_0 + \frac{\eta V^2}{4\sinh^2 R_h}\left[\cosh 2R_h - \cosh(2R_h x/d)\right]\right\},$$

where T_0 is the temperature of the electrically non-conducting wall which is a distance d from the fixed wall and moves, parallel to itself, with velocity V. Hence show that the temperature of the thermally insulated wall always exceeds T_0 and is independent of the Hartmann number R_h, but that when $R_h \gg 1$ the heat flux at the moving wall is proportional to R_h.

7. Consider Couette flow between insulating planes a distance d apart with a magnetic field of strength B_0 attached to the fixed plane $x = 0$ and directed normal to its surface. Assuming that there is no externally imposed electric field, determine the transverse current density $j_y(x)$ that flows when the plane $x = d$ moves, parallel to itself, with velocity V in the positive z-direction. Find the force that is transmitted through the magnetic field to the lower plane $x = 0$, to which the magnetic field is attached, due to the electromagnetic force $\frac{1}{c}\,\boldsymbol{j}(x) \times \boldsymbol{B}_0$ that is produced when the current flows in the magnetic field. By combining this force with the viscous frictional force $\eta\,\frac{dv_z}{dx}$ at $x = 0$, where $v_z(x)$ is the fluid velocity distribution, show that the total force per unit length acting on the fixed plane has magnitude $(\eta V R_h/d) \coth R_h$ and acts in the negative z-direction. Notice that since the magnetic field is not attached to the upper insulating plane $x = d$, the pure viscous frictional force that acts on it is equal and opposite to the combined viscous and electromagnetic forces acting on the fixed plane. By defining the non-dimensional **drag coefficient** C_D for a plane by the relation $C_D =$ (total drag force per unit length)$/\frac{1}{2}\rho V^2$, where ρ is the fluid density, show that when $R_h \gg 1$

$$\tfrac{1}{2} C_D \mathscr{R} = R_h \coth R_h,$$

where $\mathscr{R} = \rho d V/\eta$ is the Reynolds number for the flow.

8. Show that in the Hartmann flow of a viscous, electrically-conducting fluid between fixed insulating walls

a distance $2d$ apart, the velocity profile for large Hartmann number R_h becomes

$$v(x) = \left(\frac{k_1 d^2}{\eta R_h}\right)\{1-\exp\left[-R_h(1-|x|/d)\right]\}.$$

In this expression k_1 is the pressure gradient along the channel, η is the coefficient of viscosity and x is a distance measured from the centre of the channel in a direction normal to the walls. Assuming that a boundary layer of thickness d/R_h occurs at each wall find the proportion of the fluid flow that occurs outside the boundary layers.

CHAPTER IV

WAVES AND THE THEORY OF CHARACTERISTICS

§ 23. **Definitions and basic ideas.** In order that we may discuss the propagation of magnetohydrodynamic disturbances more generally than in our earlier brief examination of Alfvén waves, we must first examine some of the fundamental properties that are shared by all partial differential equations describing wave motion. Let us begin by defining a **wave** in a medium to be a disturbance in the medium which propagates with a finite speed into a known state (constant or non-constant). This definition immediately implies that there must be an advancing **wavefront** separating the region through which the disturbance has passed from the region it is about to enter. If, now, we consider the simple case of a wave advancing into a region of constant state, it is easy to see that a discontinuity in the solution must occur across the wavefront. To show this we need only notice that ahead of the wavefront the solution is constant, whilst behind it the solution is, in general, non-constant and satisfies the set of differential equations describing the motion. Recalling equation (13.1) we see that when the solution itself is continuous across the wavefront and the singularity at the wavefront is confined to a discontinuity of the derivatives of the solution, then the wavefront is a **surface of weak discontinuity.**

We shall choose to adopt this as our mathematical definition of a wavefront and will use it to determine the behaviour of the wavefront itself and of the solution behind the wavefront. When the discontinuities occur in the first order derivatives of the dependent variables, the surfaces showing the variation of the dependent variables with x and t experience a discontinuity in slope across the wavefront (see Fig. 11(a)). However, if the discontinuities occur in the higher order derivatives then these surfaces will appear to be smooth.

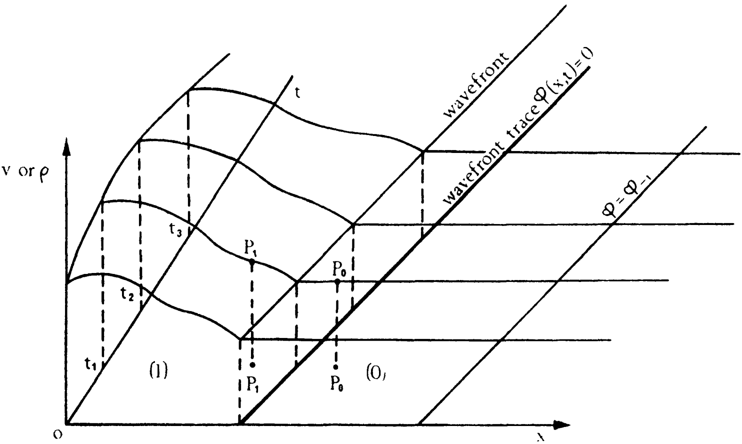

FIG. 11(a).

If, alternatively, the solution itself experiences a discontinuity across the wavefront, then a different type of wave will result. The waves corresponding to these strong discontinuities are called shock waves. We shall postpone the discussion of shock waves until Chapter VI.

Since the magnetohydrodynamic equations as typified by the Lundquist equations of § 9 involve ρ, S, $\boldsymbol{v}$ and $\boldsymbol{H}$ as dependent variables, it can be anticipated that the system of equations describing the behaviour of magnetohydrodynamic wavefronts will be rather more complicated than the corresponding equations describing wavefronts in ordinary fluid dynamics. However, since the ordinary fluid dynamic equations may be considered as a limiting form of the magnetohydrodynamic equations, we shall take advantage of their simplicity to introduce the main mathematical arguments that we shall use.

Consider the particularly simple problem of the one-dimensional time dependent **isentropic flow** of an ordinary polytropic gas described by the one-dimensional forms of the continuity equation (6.5) and of the equation of motion (6.4) without the electromagnetic force term. Taking x as the independent space coordinate and $v(x, t)$ as the fluid velocity component in the x-direction at time t, the continuity equation (6.5) becomes

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial v}{\partial x} + v \frac{\partial \rho}{\partial x} = 0. \tag{23.1}$$

Since the gas is polytropic and the flow is assumed to be isentropic it follows at once from equations (8.11) and (8.12) that the pressure p is given by

$$p = A\rho^{\gamma}, \tag{23.2}$$

where A = constant. Now, $\operatorname{grad} p = \partial p/\partial x = \left(\frac{dp}{d\rho}\right)\left(\frac{\partial \rho}{\partial x}\right)$ and so, using equation (10.19), $\operatorname{grad} p = a^2(\partial \rho/\partial x)$, where a is the velocity of sound in the gas and is, in general, a function of x and t. Thus the equation of motion (6.4) can be written

$$\rho\left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x}\right) + a^2 \frac{\partial \rho}{\partial x} = 0. \tag{23.3}$$

Equations (23.1) and (23.3) together with the constitutive equation (23.2) describe ordinary one-dimensional isentropic gas flow and will be taken as our starting point.

The form in which these equations are written is not immediately helpful for our discussion since the relationship between the wavefront and these equations is not at once apparent. To overcome this difficulty we shall change from the independent variables x and t to new independent variables ϕ and t', which we shall choose so that the curve $\phi = 0$ bears a special relationship to the wavefront. In order that we may see how best to choose ϕ let us now consider Fig. 11(*a*) in which the ordinate above the (x, t)-plane may be taken to be either $v(x, t)$ or $\rho(x, t)$. If, to simplify the diagram, we assume that the wave is advancing into a constant state, then the solution behind the wavefront and above the region (1) in the (x, t)-plane is non-constant, whereas the solution ahead of the wavefront and above the region (0) is constant.

The curves drawn at times 0, t_1, t_2 and t_3 show, qualitatively, the changing values of the dependent variables v or ρ as functions of points $P_1(x, t)$ chosen to lie in region (1) of the (x, t)-plane. The values of v and ρ corresponding to all points P_0 chosen to lie in the region (0) of the (x, t)-plane are of course constant. Since the wave is assumed to be advancing, the position x of the wavefront at any specific time is a function of the time t that has elapsed since the start of the motion. Geometrically this function appears as the projection of the wavefront onto the (x, t)-plane and it is shown in Fig. 11(*b*) as the thick line in the (x, t)-plane that passes through the point $(x_1, 0)$. We shall call this line the **wavefront trace** on the (x, t)-plane, though in current literature it is often loosely referred to as the wavefront itself.

Since the discontinuity in the derivatives of v and ρ only appear when crossing the wavefront we shall use the curves $\phi(x, t) =$ constant, where the curve $\phi(x, t) = 0$ is taken to coincide with the wavefront trace, as one of the two families of curvilinear coordinate lines with which we shall replace the $x =$ constant and $t =$ constant net. Varying the time while holding ϕ constant then simply moves a reference point P along the curve $\phi =$ constant. Hence, since (in these coordinates) the wavefront trace can never be crossed by only changing the time, limiting operations involving time alone can never lead to discontinuities in the derivatives

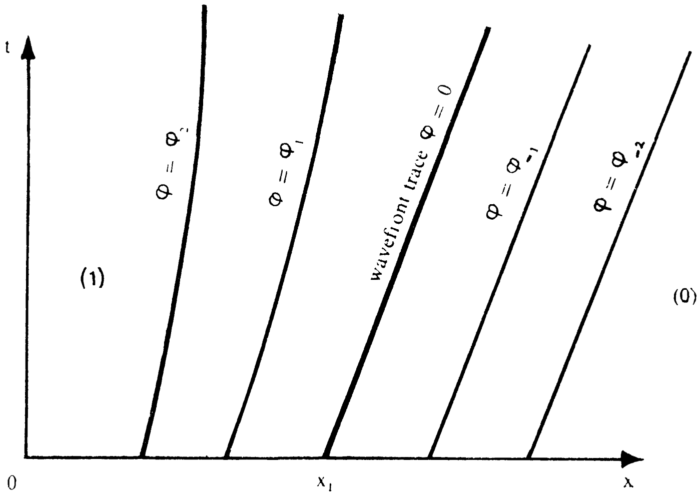

FIG. 11(*b*).

of v and ρ with respect to time. Time is thus still a convenient coordinate variable to use in conjunction with ϕ as the other coordinate variable and so we shall leave time unchanged.

Consequently we shall take the values of ϕ and of the time as our new independent variables. To avoid confusion in the subsequent manipulation we shall denote the time in our new coordinate system by t', using the fact that

$$t' = t. \tag{23.4}$$

In terms of ϕ and t' the operators $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$ become

$$\frac{\partial}{\partial t} \equiv \frac{\partial \phi}{\partial t}\frac{\partial}{\partial \phi} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} \quad \text{and} \quad \frac{\partial}{\partial x} \equiv \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'}$$

or, because of the choice of the coordinate t' in (23.4),

$$\frac{\partial}{\partial t} \equiv \frac{\partial \phi}{\partial t}\frac{\partial}{\partial \phi} + \frac{\partial}{\partial t'} \text{ and } \frac{\partial}{\partial x} \equiv \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}. \tag{23.5}$$

Equations (23.1) and (23.3) can then be written as

$$\frac{\partial \rho}{\partial t'} + \left(\frac{\partial \phi}{\partial t}\right)\left(\frac{\partial \rho}{\partial \phi}\right) + \left(\frac{\partial \phi}{\partial x}\right)\left(\rho\frac{\partial v}{\partial \phi} + v\frac{\partial \rho}{\partial \phi}\right) = 0 \tag{23.6}$$

and

$$\frac{\partial v}{\partial t'} + \left(\frac{\partial \phi}{\partial t}\right)\left(\frac{\partial v}{\partial \phi}\right) + \left(\frac{\partial \phi}{\partial x}\right)\left(v\frac{\partial v}{\partial \phi} + \frac{a^2}{\rho}\frac{\partial \rho}{\partial \phi}\right) = 0, \tag{23.7}$$

respectively.

However along the curves $\phi(x, t) =$ constant the total differential of ϕ is zero, and so $\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial t}dt = 0$ or, setting $\frac{dx}{dt} = \lambda$, this becomes

$$\frac{dx}{dt} = -\left(\frac{\partial \phi}{\partial t}\right)\bigg/\left(\frac{\partial \phi}{\partial x}\right) = \lambda. \tag{23.8}$$

An inspection of Fig. 11(*b*) will show that λ is the gradient of the $\phi =$ constant curves in the (x, t)-plane and so, on $\phi = 0$, corresponds to the **wavefront propagation speed.** At points behind the wavefront trace the appropriate values of λ represent the propagation speeds of the continuous wave located at those points.

If we denote the jump in a quantity X across the wavefront by $[X]$, then by taking points P_0 and P_1 on opposite sides of and arbitrarily close to the wavefront trace $\phi = 0$ we have that $[X] \equiv X(P_0) - X(P_1)$. Since, by our choice of coordinates, derivatives with respect to t' suffer no discontinuity when differenced in this manner we at once see that $\left[\frac{\partial \rho}{\partial t'}\right] = \left[\frac{\partial v}{\partial t'}\right] \equiv 0$. Thus, differencing equation

(23.6) across the wavefront trace in this way, we obtain the equation

$$\left(\frac{\partial \phi}{\partial t}\right)\left[\frac{\partial \rho}{\partial \phi}\right]+\left(\frac{\partial \phi}{\partial x}\right)\left(\rho\left[\frac{\partial v}{\partial \phi}\right]+v\left[\frac{\partial \rho}{\partial \phi}\right]\right)=0,$$

while equation (23.7) when similarly differenced becomes

$$\left(\frac{\partial \phi}{\partial t}\right)\left[\frac{\partial v}{\partial \phi}\right]+\left(\frac{\partial \phi}{\partial x}\right)\left(v\left[\frac{\partial v}{\partial \phi}\right]+\frac{a^2}{\rho}\left[\frac{\partial \rho}{\partial \phi}\right]\right)=0.$$

Dividing these equations by $\left(\frac{\partial \phi}{\partial x}\right)$ and using equations (23.8) we finally obtain the homogeneous simultaneous equations

$$(v-\lambda)\left[\frac{\partial \rho}{\partial \phi}\right]+\rho\left[\frac{\partial v}{\partial \phi}\right]=0 \tag{23.9}$$

and

$$\frac{a^2}{\rho}\left[\frac{\partial \rho}{\partial \phi}\right]+(v-\lambda)\left[\frac{\partial v}{\partial \phi}\right]=0. \tag{23.10}$$

Now it is well known † that these equations can only have a non-trivial solution allowing discontinuities to exist (i.e., $\left[\frac{\partial \rho}{\partial \phi}\right] \neq 0$, $\left[\frac{\partial v}{\partial \phi}\right] \neq 0$) when the determinant of their coefficients vanishes, and so

$$\begin{vmatrix} (v-\lambda) & \rho \\ \dfrac{a^2}{\rho} & (v-\lambda) \end{vmatrix}=0. \tag{23.11}$$

The equation is called the **characteristic determinant** which, as we shall see shortly, when used in conjunction with equation (23.8), defines two sets of characteristic curves

† See Aitken, *Determinants and Matrices*, 1954, § 28.

with important properties. Expanding the characteristic determinant then gives the **characteristic relation**

$$(v-\lambda)^2 - a^2 = 0, \tag{23.12}$$

which implies the following permissible values of λ,

$$\lambda = v \pm a. \tag{23.13}$$

Now, even with the small amount of information we have so far obtained, we are able to deduce some interesting properties of isentropic flow. Since λ represents the propagation speed at any point we see first that the wavefront corresponding to $\phi = 0$ and the continuous disturbance existing behind it all propagate either forwards or backwards with a speed which is the algebraic sum of the fluid speed v and the local speed of sound a. In particular, when the wave is advancing into a constant state, since v and a are continuous across the wavefront, λ is constant, and so the wavefront trace will be a straight line as shown in Fig. 11(b) with its gradient equal to the propagation speed $v_{\text{const}} \pm a_{\text{const}}$. Since we have not specifically used the assumption of a constant state ahead of the wave it is at once apparent that these ideas are quite general and extend immediately to waves advancing into a non-constant state.

The curves ϕ = constant are determined by combining equations (23.8) and (23.13) to obtain the two **equations of the characteristics**

$$\frac{dx}{dt} = v + a \tag{23.14a}$$

and

$$\frac{dx}{dt} = v - a \tag{23.14b}$$

respectively, which must be integrated to find x as a function of t.

The actual value assigned to each of these ϕ = constant curves in either of the two families (23.14a, b) is usually immaterial since ϕ does not generally enter explicitly into the analysis. However, should it be necessary to assign a value ϕ to each of the curves belonging to a family it may easily be achieved as follows. If, at time $t = 0$, the wavefront is located at $x = x_1$, define a single valued function of x, say $f(x)$, along the x-axis such that $f(x_1) = 0$ and then assign to the $\phi =$ constant curve of the family passing through the point $(x, 0)$ the value $f(x)$. For a wave starting at $x = 0$ at time $t = 0$ it is often convenient to set $f(x) = x$, which is equivalent to setting $\phi(x, 0) = x$.

We should notice that since v and a are only known when the solution has been found we are not immediately able to use equations (23.14*a*, *b*) to determine x as a function of t. We can, however, use them to determine the wavefront trace since, as we have already remarked, v and a are continuous across the wavefront and so on the wavefront can be assigned their known values (constant or non-constant) ahead of the wavefront. Equation (23.14*a*) describes advancing waves whilst equation (23.14*b*) describes receding waves. The integral curves corresponding to equations (23.14*a*, *b*) are called the **characteristic curves** of the equations (23.1) and (23.3). The family (23.14*a*) describing the advancing waves is sometimes denoted by $C^{(+)}$ and the family corresponding to equation (23.14*b*) by $C^{(-)}$.†

The arguments used in deriving the characteristic relation are of fundamental importance and, in fact, form the basis of classification of all partial differential equations. When, as happened here, the roots of the characteristic relation are real, the equations are called **hyperbolic** and, as our interpretation of λ shows, describe wave propagation at a finite speed. Had the roots been imaginary there would have been no real families of characteristics $C^{(+)}$ and the partial differential equations from which the characteristic relation

† It is possible to introduce characteristic curves in another manner and this is indicated in Example 1 of § 28.

had been derived would then have been called **elliptic.** Although we have only applied these ideas to first order equations they can easily be extended to higher order equations showing, for example, that Laplace's equation is elliptic whereas the heat equation is **parabolic**, corresponding to the coincidence of the $C^{(+)}$ and $C^{(-)}$ families of characteristics.

In our simple example there were only the two families of characteristics $C^{(\pm)}$, but it is easy to see that in a more general system of first order partial differential equations involving n dependent variables there will, in general, be n families of characteristics $C^{(1)}, C^{(2)}, \ldots, C^{(n)}$ (see Example 3, § 28). We shall see this very clearly when we proceed to discuss magnetohydrodynamic waves.

The fact that we cannot immediately integrate the characteristic equations (23.14a, b) is a direct consequence of their non-linearity. Had the governing partial differential equations been linear, as for example in the description of the propagation of infinitesimal disturbances like sound waves, the $C^{(\pm)}$ families of characteristics could have been found immediately (see Example 5, § 28).

Although so far we have only applied these ideas to one space dimension, they do in fact apply equally well to two and three space dimensions. This can be shown either by using the same type of argument that has just been outlined, with the characteristic curves $\phi(x, t) =$ constant replaced by the **characteristic surfaces** $\phi(x, y, z, t) =$ constant (see Examples 4 and 5, § 28), or by choosing the x-axis to be normal to some element P_0 of the wavefront at a given instant of time. For then, in the neighbourhood of P_0 in both space and time, the problem is locally one-dimensional and of the form examined here. This latter approach is particularly useful when the wavefront is plane, for then it is unnecessary to determine the behaviour along the normal to each element of the wavefront with advancing time.

§ **24. Rays and characteristic surfaces.** Since λ is the normal velocity of propagation of the wavefront, the following simple geometrical method of construction for a two or three-dimensional wavefront is often useful. At each point P_0 of the wavefront S_0 at time t_0 construct a vector, normal to S_0, of length $\lambda(P_0)\delta t$, where δt is a small increment of time. The wavefront S at time $t_0+\delta t$ is then

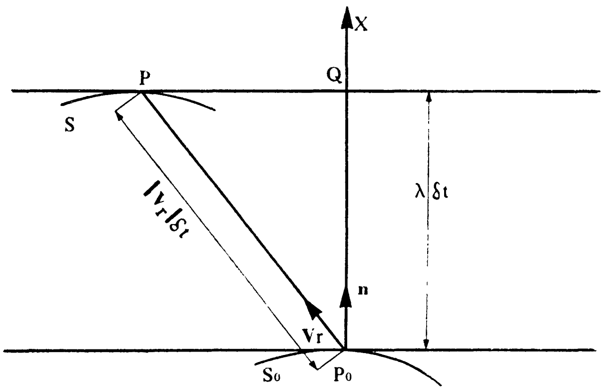

FIG. 12.

approximated by the envelope of all the planes which are normal to and located on the end points of each such vector. This construction generates the new wavefront from elements of plane waves and is essentially similar to the construction often used in geometrical optics (first suggested by Huygens), which uses instead the envelope of all the spherical wavefronts emanating from point sources of light on a given wavefront.

This analogy with geometrical optics can be pursued a little further if we introduce the idea of a ray. To do this consider Fig. 12 in which S_0 represents an element of a wavefront at time t_0 which, in time increment δt, moves with velocity $\boldsymbol{v}_r$ to become surface element S, so that a point P_0 of S_0 becomes point P of S.

In general the velocity $\boldsymbol{v}_r$ of the point P_0 of S_0 will not be directed along the normal $\boldsymbol{n}$ to S_0 at P_0. However, since by our construction the tangent plane to S, taken parallel to the tangent plane at P_0, must form part of the envelope of the wavefront at time $t_0+\delta t$, the distance P_0Q measured in the x-direction along $\boldsymbol{n}$ must be $\lambda(P_0)\delta t$. Consequently, if we define the velocity $\boldsymbol{v}_r$ of the point P_0 to be the **ray velocity** at P_0, we at once see that the relationship between the ray velocity $\boldsymbol{v}_r$ at P_0 and the normal $\boldsymbol{n}$ is simply

$$\lambda = \boldsymbol{n} \,.\, \boldsymbol{v}_r. \tag{24.1}$$

The vector $\overline{P_0P}$ is called the **ray** vector through point P_0.

In a continuum medium the ray velocity can be interpreted as the velocity of the particles comprising the wavefront and, as we have just indicated, is distinct from the normal velocity. For this reason the wavefront is sometimes called the **surface of normal velocity** to emphasise that such a surface does not take into account the way in which the individual points comprising the surface transform with time.

We shall postpone discussing how the characteristic curves which we have just introduced may be used to help determine the flow conditions behind a wavefront until we come to discuss the special case of magnetohydrodynamic simple waves. Example 6 of § 28 does, however, indicate how they may be used in the special case of ordinary isentropic flow.

Before examining the characteristic curves that occur in magnetohydrodynamics let us return briefly to the idea of **characteristic equations.** Either of the two characteristic equations displayed in (23.9) or (23.10) may be used to determine the relationship that must exist between $\left[\dfrac{\partial \rho}{\partial \phi}\right]$

and $\left[\frac{\partial v}{\partial \phi}\right]$ in one-dimensional isentropic flow when use is made of the permissible values of λ given in (23.13). We obtain the results

$$-a\left[\frac{\partial \rho}{\partial \phi}\right]+\rho\left[\frac{\partial v}{\partial \phi}\right]=0 \text{ along } C^{(+)} \text{ characteristics} \tag{24.2a}$$

and

$$a\left[\frac{\partial \rho}{\partial \phi}\right]+\rho\left[\frac{\partial v}{\partial \phi}\right]=0 \text{ along } C^{(-)} \text{ characteristics.} \tag{24.2b}$$

When the wave is advancing into a constant state these equations have a convenient interpretation in terms of the infinitesimal increments experienced by ρ and v themselves in the vicinity of the wavefront $\phi = 0$. To establish this we shall use the obvious fact that when propagating into a constant state, all derivatives with respect to ϕ ahead of the wavefront will vanish and so the jump $\left[\frac{\partial X}{\partial \phi}\right]$ that is experienced by a function X of ϕ and t' when crossing $\phi = 0$ reduces to $\frac{\partial X}{\partial \phi}$, evaluated immediately behind the wavefront. If, now, we approximate $\frac{\partial X}{\partial \phi}$ behind the wavefront $\phi = 0$ at time t_0' by the difference equation

$$\frac{\partial X}{\partial \phi}=\frac{X(0, t_0')-X(\delta\phi_0, t_0')}{\delta\phi_0}, \tag{24.3}$$

where $\phi = \delta\phi_0$ is a characteristic curve close behind the wavefront, we can then write, approximately,

$$\frac{\partial X}{\partial \phi}=\frac{\delta X}{\delta \phi_0}, \tag{24.4}$$

with $\delta X = X(0, t_0')-X(\delta\phi_0, t_0')$.

Applying this result to the characteristic equations (24.2*a*, *b*) then gives the alternative form of the characteristic equations

$$-a\delta\rho+\rho\delta v = 0 \text{ along } C^{(+)} \text{ characteristics} \tag{24.5a}$$

and

$$a\delta\rho+\rho\delta v = 0 \text{ along } C^{(-)} \text{ characteristics.} \tag{24.5b}$$

This notation has the advantage that it correctly suggests that when wave propagation is into a constant state the $\delta\rho$ and δv of equations (24.5*a*, *b*) can be replaced by the differentials $d\rho$ and dv which are, of course, infinitesimals. However, for wave propagation into a non-constant state, the δ notation must be regarded as an alternative to our earlier notation representing the jump in the derivative of the associated variable normal to the wavefront. The jumps in the derivatives are finite but can be arbitrarily large, subject only to the requirement they be compatible with the characteristic equations.

§ 25. Magnetohydrodynamic characteristic equations. We now consider the application of the ideas contained in §§ 23 and 24 to magnetohydrodynamic flows. However, before doing so we first simplify the problem by assuming that dissipative effects may be neglected and that the fluid is polytropic. The consequences of these assumptions have already been examined in § 9 where it was shown that they result in the Lundquist equations

$$\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\boldsymbol{v}) = 0, \tag{25.1}$$

$$\frac{\partial\boldsymbol{v}}{\partial t}+(\boldsymbol{v}\,.\,\text{grad})\boldsymbol{v}-\frac{\mu}{4\pi\rho}(\text{curl}\,\boldsymbol{H})\times\boldsymbol{H}+\text{grad}\,p(\rho, S)=\boldsymbol{0}, \tag{25.2}$$

$$\frac{\partial S}{\partial t}+(\boldsymbol{v}\,.\,\text{grad})S = 0, \tag{25.3}$$

$$\frac{\partial \boldsymbol{H}}{\partial t} - \text{curl}\,(\boldsymbol{v} \times \boldsymbol{H}) = 0, \tag{25.4}$$

$$\text{div}\, \boldsymbol{H} = 0, \tag{25.5}$$

together with the polytropic gas law

$$p = A(S)\rho^{\gamma}. \tag{25.6}$$

If we wished we could now express these equations in component form and, by proceeding in a manner strictly analogous to that employed in the previous two sections (see also Example 4, § 28), we could derive the characteristic determinant and characteristic curves appropriate to the Lundquist equations. This method, although perfectly satisfactory, requires considerable algebraic manipulation. Fortunately this can be avoided if we make use of the argument presented at the end of § 23 and orient the x-axis so that it is directed normal to the wavefront surface at some given point P of interest. For then, in the neighbourhood of this point, the problem is locally one-dimensional and we may simplify equations (25.1) to (25.5) by considering only their one-dimensional form in (x, t)-space.

The gradient term in equation (25.2) can be simplified by using the result

$$\text{grad}\, p(\rho, S) = \left(\frac{\partial p}{\partial \rho}\right) \text{grad}\, \rho + \left(\frac{\partial p}{\partial S}\right) \text{grad}\, S,$$

which combined with equation (10.19) gives

$$\text{grad}\, p(\rho, S) = a^2\, \text{grad}\, \rho + \left(\frac{\partial p}{\partial S}\right) \text{grad}\, S, \tag{25.7}$$

where a is the local speed of sound. This then allows us to write the one-dimensional Lundquist equations in the form:

Continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_x)}{\partial x} = 0,$$

Momentum equation

$$\frac{\partial \boldsymbol{v}}{\partial t}+v_x\frac{\partial \boldsymbol{v}}{\partial x}-\frac{\mu}{4\pi\rho}\left(\boldsymbol{n}\times\frac{\partial \boldsymbol{H}}{\partial x}\right)\times\boldsymbol{H}+\frac{1}{\rho}\left(a^2\frac{\partial\rho}{\partial x}+\left(\frac{\partial p}{\partial S}\right)\left(\frac{\partial S}{\partial x}\right)\right)\boldsymbol{n}=0,$$

Magnetic induction equation

$$\frac{\partial \boldsymbol{H}}{\partial t}-\boldsymbol{n}\times\frac{\partial(\boldsymbol{v}\times\boldsymbol{H})}{\partial x}=0,$$

Energy equation (isentropic condition)

$$\frac{\partial S}{\partial t}+v_x\frac{\partial S}{\partial x}=0,$$

Solenoidal condition

$$\frac{\partial H_x}{\partial x}=0,$$

where $\boldsymbol{n}$ is a unit vector which is normal to the wavefront and directed along the positive x-axis. When written in component form these equations become

$$\frac{\partial\rho}{\partial t}+v_x\frac{\partial\rho}{\partial x}+\rho\frac{\partial v_x}{\partial x}=0, \tag{25.8}$$

$$\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+\frac{\mu}{4\pi\rho}H_y\frac{\partial H_y}{\partial x}+\frac{\mu}{4\pi\rho}H_z\frac{\partial H_z}{\partial x}+\frac{a^2}{\rho}\frac{\partial\rho}{\partial x}+\frac{1}{\rho}\left(\frac{\partial p}{\partial S}\right)\left(\frac{\partial S}{\partial x}\right)=0, \tag{25.9}$$

$$\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}-\frac{\mu}{4\pi\rho}H_x\frac{\partial H_y}{\partial x}=0, \tag{25.10}$$

$$\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}-\frac{\mu}{4\pi\rho}H_x\frac{\partial H_z}{\partial x}=0, \tag{25.11}$$

$$\frac{\partial H_x}{\partial t} = 0, \tag{25.12}$$

$$\frac{\partial H_y}{\partial t} + \frac{\partial}{\partial x}(H_y v_x - H_x v_y) = 0, \tag{25.13}$$

$$\frac{\partial H_z}{\partial t} + \frac{\partial}{\partial x}(H_z v_x - H_x v_z) = 0, \tag{25.14}$$

$$\frac{\partial S}{\partial t} + v_x \frac{\partial S}{\partial x} = 0 \tag{25.15}$$

$$\frac{\partial H_x}{\partial x} = 0, \tag{25.16}$$

where the suffixes, x, y and z are used to denote the appropriate vector components.

Now equations (25.12) and (25.16) show that in the neighbourhood of point P of the wavefront, $H_x = H_{0x}$ is constant. Thus, since the wavefront is a surface of weak discontinuity, and so $\boldsymbol{H}$, $\boldsymbol{v}$, ρ and S are continuous across the wavefront, it immediately follows that $H_x = H_{0x}$ is a constant across the entire wavefront. Using the notation of § 24 this result becomes

$$\delta H_x = 0. \tag{25.17}$$

The remaining seven equations then describe the magnetohydrodynamic flow that occurs both immediately in front of and behind the wavefront. So far, although we have chosen the x-axis in a convenient manner, we have left undetermined the precise orientation of the mutually orthogonal y and z-axes. Let us now take advantage of this fact to simplify our equations still further by choosing the z-axis so that H_z, the z-component of the magnetic field behind the wavefront, becomes zero. When H_z is set equal to zero in equations (25.8) to (25.14) and the arguments of

§§ 23 and 24 are used they yield the following set of **magneto-hydrodynamic characteristic equations**

$$\mp c_n \delta\rho + \rho \delta v_x = 0, \tag{25.18}$$

$$\mp c_n \rho \delta v_x + a^2 \delta\rho + \left(\frac{\partial p}{\partial S}\right) \delta S + \frac{\mu}{4\pi} H_y \delta H_y = 0, \tag{25.19}$$

$$\mp c_n \rho \delta v_y - \frac{\mu}{4\pi} H_x \delta H_y = 0, \tag{25.20}$$

$$\mp c_n \rho \delta v_z - \frac{\mu}{4\pi} H_x \delta H_z = 0, \tag{25.21}$$

$$\mp c_n \delta H_y + H_y \delta v_x - H_x \delta v_y = 0, \tag{25.22}$$

$$\mp c_n \delta H_z - H_x \delta v_z = 0, \tag{25.23}$$

$$\mp c_n \delta S = 0, \tag{25.24}$$

where, if the velocity λ of the wavefront trace $\phi(x, t) = 0$ is determined by $\lambda = -\left(\frac{\partial\phi}{\partial t}\right) \Big/ \left(\frac{\partial\phi}{\partial x}\right)$, then

$$c_n = |\lambda - v_x| \tag{25.25}$$

is the modulus of the speed of the wavefront relative to the fluid. The minus and plus signs associated with c_n in these equations correspond, respectively, to the negative and positive values of $v_x - \lambda$.

When $c_n \neq 0$, equation (25.24) implies that

$$\delta S = 0. \tag{25.26}$$

If we set $\delta S = 0$ in equations (25.18) to (25.23) and arrange the terms in the equations in the order $\delta\rho$, δv_x, δv_y, δv_z,

δH_y and δH_z it is easy to see that the **magnetohydrodynamic characteristic determinant**

$$\begin{vmatrix} \mp c_n & \rho & 0 & 0 & 0 & 0 \\ \dfrac{a^2}{\rho} & \mp c_n & 0 & 0 & \dfrac{\mu H_y}{4\pi\rho} & \dfrac{\mu H_z}{4\pi\rho} \\ 0 & 0 & \mp c_n & 0 & -\dfrac{\mu H_x}{4\pi\rho} & 0 \\ 0 & 0 & 0 & \mp c_n & 0 & -\dfrac{\mu H_x}{4\pi\rho} \\ 0 & H_y & -H_x & 0 & \mp c_n & 0 \\ 0 & H_z & 0 & -H_x & 0 & \mp c_n \end{vmatrix} = 0 \quad (25.27)$$

is the condition that these homogeneous equations should possess a non-trivial solution.

It is convenient to regard c_n rather than λ as the fundamental parameter in this characteristic determinant and in the associated characteristic relation. If we define the **Alfvén speed** b by the expression

$$b = \sqrt{\frac{\mu H^2}{4\pi\rho}}, \qquad (25.28)$$

and the Alfvén speed b_x in the x-direction by the expression

$$b_x = \sqrt{\frac{\mu H_{0x}^2}{4\pi\rho}}, \qquad (25.29)$$

the **magnetohydrodynamic characteristic relation** derived from (25.27) becomes

$$(c_n^2 - b_x^2)\{(c_n^2 - a^2)(c_n^2 - b_x^2) - c_n^2(b^2 - b_x^2)\} = 0. \quad (25.30)$$

This equation has the characteristic roots

$$c_n = b_x, \tag{25.31a}$$

$$c_n = c_f = \left(\tfrac{1}{2}\{(a^2+b^2)+\sqrt{(a^2+b^2)^2-4a^2b_x^2}\}\right)^{\frac{1}{2}} \tag{25.31b}$$

and

$$c_n = c_s = \left(\tfrac{1}{2}\{(a^2+b^2)-\sqrt{(a^2+b^2)^2-4a^2b_x^2}\}\right)^{\frac{1}{2}}. \tag{25.31c}$$

By using the obvious inequality

$$(a^2+b^2)^2-4a^2b_x^2 > (a^2-b^2)^2 > 0, \tag{25.32}$$

it follows that the roots c_f and c_s of equations (25.31*b*, *c*) are always real, as are the characteristic roots λ given by

$$\lambda = v_x \pm c_f, \tag{25.32a}$$

$$\lambda = v_x \pm c_s \tag{25.32b}$$

and

$$\lambda = v_x \pm b_x. \tag{25.32c}$$

The + and − signs of c_f correspond, respectively, to the − and + signs of c_n in the characteristic equations, and similarly for the root c_s. It is obvious that $c_f > c_s$. For this reason the wave corresponding to equation (25.32*a*) is called the **fast wave** and the wave corresponding to equation (25.32*b*) is called the **slow wave.** For reasons that will be presented in the next section the wave corresponding to equation (25.32*c*) is called the **transverse wave.**

Finally, returning to equation (25.24), we see that when $c_n = 0$ (i.e., $\lambda - v_x = 0$) it is possible that $\delta S \neq 0$ and so a disturbance in entropy may propagate. This disturbance, called an **entropy wave**, corresponding to the root $c_n = 0$ is characterised by the equivalent equation

$$\lambda = v_x. \tag{25.32d}$$

There are thus seven sets of characteristic curves that occur in magnetohydrodynamics corresponding to the families of solutions to the equation $\frac{dx}{dt} = \lambda$, in which λ takes one of the seven values contained in equations (25.32*a*) to (25.32*d*).

§ **26. Magnetohydrodynamic waves.** We are now in a position to determine the fundamental properties of the fast, slow, transverse and entropy waves that we have just mentioned. To do this we shall make use of the magnetohydrodynamic characteristic equations (25.18) to (25.24) and the characteristic roots (25.31*a*, *b*, *c*).

(a) *Fast and slow waves*

If we now assume that $c_n \neq 0$ and that the factor $(c_n^2 - b_x^2)$ in the characteristic relation (25.30) is non-zero (i.e., that $c_n \neq b_x$), we are left with the relation

$$(c_n^2 - a^2)(c_n^2 - b_x^2) = c_n^2(b^2 - b_x^2). \tag{26.1}$$

The right-hand side of this expression is always positive and so we must either have the conditions

$$c_n^2 \geqq a^2 \text{ and } c_n^2 \geqq b_x^2 \tag{26.2a}$$

or the conditions

$$c_n^2 \leqq a^2 \text{ and } c_n^2 \leqq b_x^2. \tag{26.2b}$$

By virtue of the relationship $c_f > c_s$ we see that the conditions (26.2*a*) are equivalent to the inequalities

$$c_f \geqq a \quad \text{and} \quad c_f \geqq b_x, \tag{26.3a}$$

whilst conditions (26.2*b*) are equivalent to the inequalities

$$c_s \leqq a \quad \text{and} \quad c_s \leqq b_x. \tag{26.3b}$$

It is at once apparent from equation (26.1) that equality can only occur in these relations if $b = b_x$, which corresponds to the non-existence of a transverse magnetic field. In the limit of vanishing transverse magnetic field

$c_s \to a$, the local speed of sound, whilst $c_f \to b_x$, the Alfvén speed. This fact has resulted in the fast and slow waves also being known as **magnetoacoustic waves.**

To find the fundamental properties of these magnetoacoustic waves we now combine characteristic equations (25.21) and (25.23) to obtain

$$(c_n^2 - b_x^2)\delta H_z = 0.$$

We have assumed that $(c_n^2 - b_x^2) = 0$, and so it follows at once that $\delta H_z = 0$ and hence, from equations (25.21) or (25.23), that $\delta v_z = 0$. Since we have also assumed that $c_n = 0$, equation (25.24) shows that $\delta S = 0$. So magnetoacoustic waves have the property that

$$\delta v_z = \delta H_z = \delta S = 0. \tag{26.4}$$

The remaining four magnetoacoustic characteristic equations (25.18) to (25.20) and (25.22) can be written in a convenient form if we express the jumps $\delta\rho$, δv_x, δv_y and δH_y in terms of a dimensionless parameter characterising the jump of one of these quantities. We shall express these jumps in terms of the jump $\delta\rho$ by setting

$$\delta\rho = \varepsilon\rho \tag{26.5a}$$

where ε is our dimensionless parameter.† It follows at once that

$$\delta v_x = -\varepsilon(\mp c_n), \tag{26.5b}$$

$$\delta v_y = \frac{\mp \varepsilon c_n b_x b_y}{(c_n^2 - b_x^2)} \operatorname{sgn}(H_x H_y), \tag{26.5c}$$

$$\delta H_y = \frac{\varepsilon H_y c_n^2}{(c_n^2 - b_x^2)}, \tag{26.5d}$$

where $b_y = \sqrt{\mu H_y^2 / 4\pi\rho}$ is the Alfvén speed in the y-direction,

† Notice that different values of the parameter ε are to be assigned to the fast and slow waves.

$\operatorname{sgn}(H_x H_y) = H_x H_y / | H_x H_y |$ is a unity multiplier carrying the sign of $H_x H_y$ and where c_n can take either the value c_f or c_s.

The pressure change δp across the wavefront can be determined directly from equation (25.7). In its one-dimensional form this equation becomes

$$\frac{\partial p}{\partial x} = a^2 \frac{\partial \rho}{\partial x} + \left(\frac{\partial p}{\partial S}\right)\left(\frac{\partial S}{\partial x}\right),$$

and by applying the operator $\dfrac{\partial}{\partial x} \equiv \dfrac{\partial \phi}{\partial x} \dfrac{\partial}{\partial \phi}$ defined in (23.5) and differencing the result across the wavefront we immediately obtain

$$\delta p = a^2 \delta \rho + \left(\frac{\partial p}{\partial S}\right) \delta S. \qquad (26.6)$$

The pressure change in magnetoacoustic waves is then given by

$$\delta p = a^2 \delta \rho. \qquad (26.7)$$

(b) *Transverse waves*

We have seen that transverse waves are characterised by the condition $c_n = b_x$. Let us now also assume that c_f and c_s are both distinct from c_n. Setting $c_n = b_x$ in the magnetohydrodynamic characteristic equations immediately leads to the results

$$\delta v_x = \delta v_y = \delta H_y = \delta S = \delta \rho = 0 \qquad (26.8a)$$

and

$$\delta v_z = \mp \sqrt{\frac{\mu}{4\pi\rho}} \operatorname{sgn}(H_x) \delta H_z. \qquad (26.8b)$$

The $\mp$ sign in this equation corresponds to the $\mp$ sign associated with c_n in equations (25.22) and (25.23), respectively. Since the vector $\delta \boldsymbol{H}$ has δH_z as its only non-zero component, it is a transverse vector with respect to the wavefront normal $\boldsymbol{n}$, and so it can be expressed in terms of a dimensionless parameter ε characterising the size of the jump by means of the vector equation

$$\delta \boldsymbol{H} = \varepsilon \boldsymbol{n} \times \boldsymbol{H}. \tag{26.9a}$$

The vector jump $\delta \boldsymbol{v}$ which has δv_z as its only non-zero component may be written

$$\delta \boldsymbol{v} = \mp \varepsilon \sqrt{\frac{\mu}{4\pi\rho}} \operatorname{sgn}(H_x) \boldsymbol{n} \times \boldsymbol{H}. \tag{26.9b}$$

These results show that the change across the wavefront only occurs in the transverse components of $\delta \boldsymbol{H}$ and $\delta \boldsymbol{v}$ which is the reason for the name **transverse wave**.

Combining equations (26.6) and (26.8*a*) demonstrates that transverse waves are non-compressive, since $\delta p = 0$. We can, however, establish more than this if we consider the jump in the derivative of $\boldsymbol{H}^2$ across the wavefront. For, by differentiating $\boldsymbol{H}^2$ with respect to ϕ and differencing the result across the wavefront, it is easily shown that

$$\delta(\boldsymbol{H}^2) = 2\boldsymbol{H}.\delta\boldsymbol{H}.$$

Consequently, from the expression for $\delta \boldsymbol{H}$ in equation (26.9*a*) and the properties of a triple scalar product, we see that

$$\delta(\boldsymbol{H}^2) = 0, \tag{26.9c}$$

showing that $\delta p^* = 0$, where $p^* = p + \dfrac{\mu \boldsymbol{H}^2}{8\pi}$ is the **total pressure** (i.e., the sum of the fluid and magnetic pressures).

In § 11 we have already shown that an Alfvén wave is a transverse wave that occurs in an incompressible fluid and it must obviously be a special form of the more general transverse wave discussed here. For incompressible fluids we must modify the magnetohydrodynamic characteristic equations by setting $\delta\rho \equiv 0$ and replacing $a^2\delta\rho$ by δp in order to determine the exact relationship that exists between the jump quantities. The connection between Alfvén waves and general transverse waves is indicated in Example 10, § 28.

(c) *Entropy wave*

We have seen that an entropy wave is characterised by the root $c_n = 0$ and equation (25.32*d*) shows that the normal wave speed is v_x. Setting $c_n = 0$ in the magnetohydrodynamic equations and assuming that $H_x \neq 0$ shows that

$$\delta \boldsymbol{v} = \delta \boldsymbol{H} = 0. \tag{26.10a}$$

If we now characterise the entropy jump δS by the equation

$$\delta S = \varepsilon \tag{26.10b}$$

it at once follows that the density jump is

$$\delta \rho = -\frac{\varepsilon}{a^2}\frac{\partial p}{\partial S}. \tag{26.10c}$$

Applying these results to equation (26.6) then shows that

$$\delta p = 0. \tag{26.10d}$$

Since there is no discontinuity in velocity or pressure across an entropy wave no fluid particles can cross such a wavefront. Thus although the density and entropy undergo jumps when crossing the wavefront the fluids on adjacent sides of the wavefront do not mix. This type of wave describes the motion of two fluids that are in contact but are in two thermodynamically different states. For this reason these wavefronts are sometimes called **contact surfaces.** See Example 9 of § 28 for details of the entropy wave that occurs when $H_x = 0$.

§ **27. Magnetohydrodynamic wavefront diagrams.** The very distinctive nature of magnetohydrodynamic waves that has just been established in the discussion of the relationships that exist between the jump quantities $\delta\rho$, $\delta\boldsymbol{v}$, $\delta\boldsymbol{H}$ and δS is still further emphasised when the surfaces of normal velocity are considered. To do this we make use of equations (25.31a, b, c) and consider the manner in which waves will propagate from a fixed point source into a region of constant state.

Equations (25.31a, b, c) can be written in the form

$$\frac{b_x}{b} = \cos\theta, \tag{27.1a}$$

$$\frac{c_f}{b} = \left(\tfrac{1}{2}\{(1+s)+\sqrt{(1+s)^2-4s\cos^2\theta}\}\right)^{\frac{1}{2}} \tag{27.1b}$$

and

$$\frac{c_s}{b} = \left(\tfrac{1}{2}\{(1+s)-\sqrt{(1+s)^2-4s\cos^2\theta}\}\right)^{\frac{1}{2}}, \tag{27.1c}$$

where $s = a^2/b^2$ and θ is the angle between the wavefront normal and the magnetic field vector $\boldsymbol{H}$. We shall now regard b_x/b, c_f/b and c_s/b as radial vectors drawn at the polar angle θ from an origin located at a point P. This representation then immediately enables us to construct a polar plot of a cross-section of the three-dimensional surfaces of normal velocity. The transverse wave curve of normal velocity described by equation (27.1a) is independent of s and is easily seen to comprise two circles of unit diameter that are tangent to one another at the origin and have their centres on a line parallel to $\boldsymbol{H}$. The curves of normal velocity for the fast and slow waves corresponding to different values of s can easily be constructed from equations (27.1b, c). Points of special interest on these curves occur at $\theta = 0$, corresponding to the wavefront normal velocity along the magnetic field, and at $\theta = \pi/2$, corresponding to the wavefront normal velocity in a direction transverse to the magnetic field. Thus at $\theta = 0$, $c_f/b = (\frac{1}{2}\{(1+s)+|1-s|\})^{\frac{1}{2}}$ and $c_s/b = (\frac{1}{2}\{(1+s)-|1-s|\})^{\frac{1}{2}}$ while at $\theta = \pi/2$, $c_f/b = (1+s)^{\frac{1}{2}}$ and $c_s/b = 0$. Representative curves are shown in Fig. 13 for $s<1$ and $s>1$. The slow,

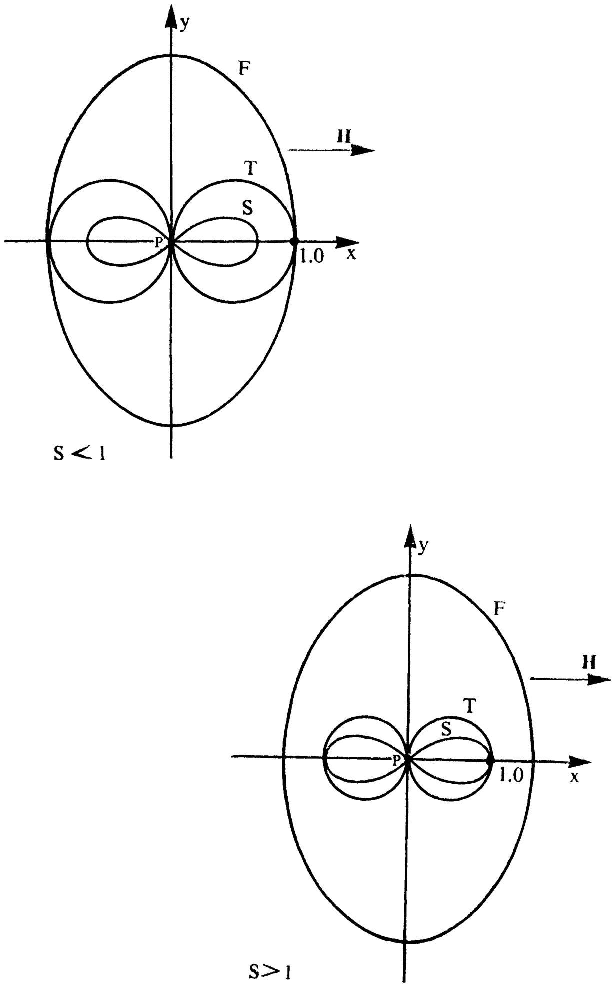

FIG. 13.

transverse and fast curves of normal velocity are denoted by the letters S, T and F, respectively. Since the curves of normal velocity constructed in this manner are the same for any plane passing through the point P located at the source and containing a line parallel to the magnetic field vector $\boldsymbol{H}$, it follows directly that the actual surfaces of normal velocity are obtained by rotating these two-dimensional curves about the x-axis.

Having determined the surfaces of normal velocity we are now in a position to use the construction described in § 24 to obtain the actual wavefronts. We recall that when the wavefront S_0 is known at a time t_0 then the wavefront S at time $t = t_0 + \delta t$ corresponds to the envelope of all the planes that are parallel to the tangent planes at all points P_0 of S_0 and are distant from the points P_0 an amount $\lambda(P_0)\delta t$ along the normals to S_0 through the respective points P_0. This construction simplifies somewhat if we assume that the flow velocity v_x in the constant state into which the waves are advancing is zero, for then

$$c_n = |\lambda - v_x| = |\lambda|$$

and c_n becomes equal to the wavefront normal velocity.

Let us now determine the shape of the fast, slow and transverse wavefronts that emanate from a fixed point source o. We start by deriving the parametric equation of a representative line belonging to the envelope in the (x, y)-plane. Since the wave is advancing into a constant state, c_n is constant, and the distance moved in a time t by a plane wavefront that is normal to a radius vector inclined at an angle θ to the x-axis (i.e., to $\boldsymbol{H}$) is $\pm c_n(\theta)t$ (see Fig. 14).

The equation of the line is

$$x \cos\theta + y \sin\theta = \pm c_n(\theta)t, \tag{27.2}$$

where the $+$ sign corresponds to a wave diverging from o and the $-$ sign corresponds to a wave converging on o.

We know from elementary analysis † that the envelope of these lines is obtained by solving equation (27.2) simultaneously with the equation

$$-x \sin \theta + y \cos \theta = \pm \left(\frac{dc_n}{d\theta}\right) t, \tag{27.3}$$

which is obtained by differentiating (27.2) with respect to θ.

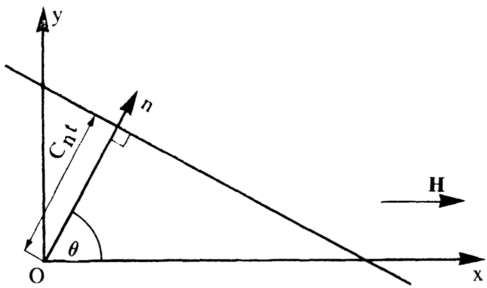

FIG. 14

The general expression for the envelope at time t is thus

$$x = \pm \left(c_n \cos \theta - \frac{dc_n}{d\theta} \sin \theta \right) t \tag{27.4a}$$

and

$$y = \pm \left(c_n \sin \theta + \frac{dc_n}{d\theta} \cos \theta \right) t. \tag{27.4b}$$

It is to be understood that the signs to be chosen in these expressions are those which make it possible to construct an envelope. The fast, slow and transverse wavefronts are then

† Gillespie, *Partial Differentiation*, 1960, pp. 66-68. The envelope of the curves $f(x, y, \alpha) = 0$ as α varies is determined by eliminating α between the two equations $f = 0$ and $\dfrac{\partial f}{\partial \alpha} = 0$.

obtained by setting c_n equal to c_f, c_s and b_x, respectively. When this is done we find that:

for the fast wave

$$\frac{x}{b} = \pm\cos\theta\left\{\frac{c_f}{b} - \frac{s\sin^2\theta}{(c_f/b)\{(1+s)^2-4s\cos^2\theta\}^{\frac{1}{2}}}\right\}t, \quad (27.5a)$$

$$\frac{y}{b} = \pm\sin\theta\left\{\frac{c_f}{b} + \frac{s\cos^2\theta}{(c_f/b)\{(1+s)^2-4s\cos^2\theta\}^{\frac{1}{2}}}\right\}t; \quad (27.5b)$$

for the slow wave

$$\frac{x}{b} = \pm\cos\theta\left\{\frac{c_s}{b} + \frac{s\sin^2\theta}{(c_s/b)\{(1+s)^2-4s\cos^2\theta\}^{\frac{1}{2}}}\right\}t, \quad (27.6a)$$

$$\frac{y}{b} = \pm\sin\theta\left\{\frac{c_s}{b} - \frac{s\cos^2\theta}{(c_s/b)\{(1+s)^2-4s\cos^2\theta\}^{\frac{1}{2}}}\right\}t; \quad (27.6b)$$

and for the transverse wave

$$x = \pm bt, \quad (27.7a)$$

$$y = 0. \quad (27.7b)$$

Since the wavefront is symmetrical about the x-axis, and the wavefront on any other plane drawn through the x-axis will be identical, it follows that the actual three-dimensional wavefronts are obtained by rotating these figures about the x-axis. Typical wavefront diagrams in the (x, y)-plane, frequently called **Friedrichs diagrams** after K. O. Friedrichs who was the first to derive them, are shown in Figs. 15, in which Fig. 15(*a*) shows their geometrical construction and their most important dimensions (see Example 13, § 28). Here again the slow wavefront is denoted by the letter S and the fast wavefront by the letter F. The points A represent the transverse wavefront which has degenerated to two points moving with speed b in opposite directions along the x-axis. The reason for this degeneracy is easily seen when the geometrical construction is applied directly to the transverse wave curves of normal velocity for, since they are both circles tangent at the origin, all the planes forming their envelopes pass through the points A. The application of this method of construction to obtain the wavefront appropriate to a disturbance propagating from a source of finite size is indicated in Example 11 of § 28.

In the next chapter we shall show how the non-linear one-dimensional equations of magnetohydrodynamics may be integrated to describe the behaviour of an important special class of non-linear wave phenomena. The fact that the Lundquist equations are non-linear implies that the **principle of superposition** † cannot be used to obtain solutions. This valuable principle, which is only applicable to linear equations, enables the construction of the solution to a general problem to be achieved by the superposition (addition) of certain simple and easily constructed solutions. However, when the disturbances involved are small and take place about some equilibrium state which we shall denote by the suffix 0, the Lundquist equations can be **linearised** so that the superposition principle will then apply.

To illustrate this idea let us assume an equilibrium state characterised by the constant values $\boldsymbol{H}_0$, ρ_0, S_0 and $\boldsymbol{v}_0 = \boldsymbol{0}$. Assume small disturbances $\boldsymbol{h}$, ρ_1 and S_1 so that

$$\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{h}, \quad \rho = \rho_0 + \rho_1, \quad S = S_0 + S_1,$$

and assume also that the velocity $\boldsymbol{v}$ and the current $\boldsymbol{j}$ that are produced are both small. Then, regarding terms of order $\boldsymbol{h}^2$ and $|\boldsymbol{h}|.|\boldsymbol{v}|$, ..., etc., as negligible, equation (5.1′) for the magnetic field becomes

$$\frac{\partial \boldsymbol{h}}{\partial t} = \text{curl}\,(\boldsymbol{v} \times \boldsymbol{H}_0), \tag{27.8a}$$

while the equation of mass conservation (6.5) becomes

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \,\text{div}\, \boldsymbol{v} = 0. \tag{27.8b}$$

† See Coulson, *Waves*, 1949, §§ 6 and 7.

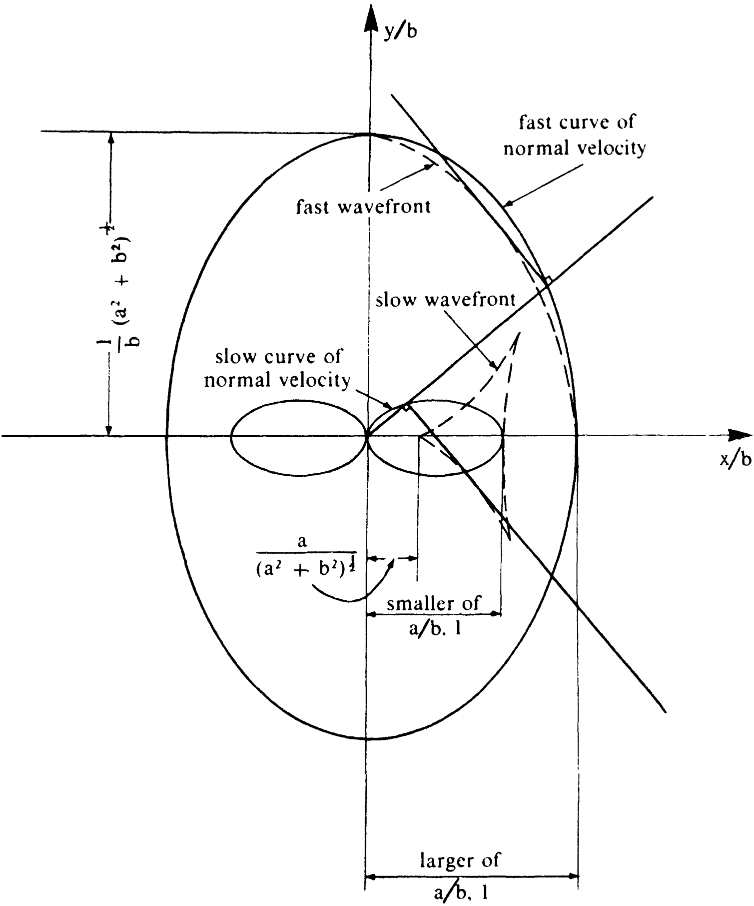
y/b
fast curve of
normal velocity
fast wavefront
slow wavefront
slow curve of
normal velocity
x/b
smaller of
a/b, 1
larger of
a/b, 1

FIG. 15 (*a*)

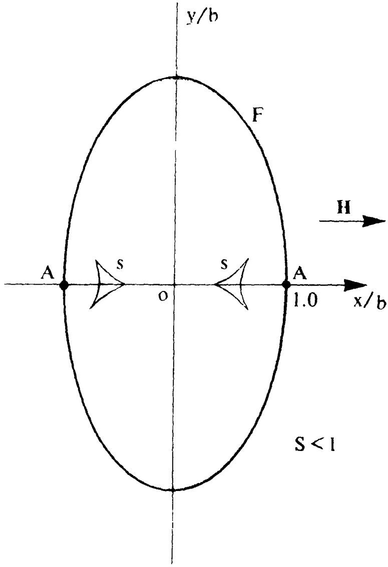

FIG. 15 (*b*)

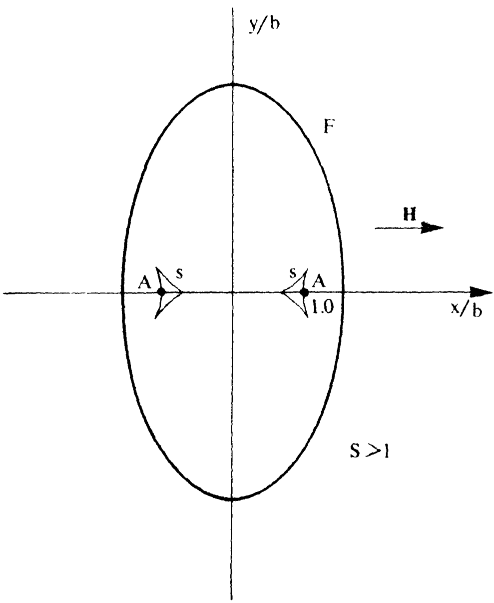

FIG. 15 (*c*)

A simple dimensional analysis of the entropy equation (8.2′) shows that the entropy is constant to first order provided $|\boldsymbol{v}| << V$, where $V = LT^{-1}$ is a characteristic velocity of the fluid. Consequently, since we are assuming that $\boldsymbol{v}$ is small, the flow is isentropic and the momentum equation (7.5′) can be written

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} + a_0^2 \operatorname{grad} \rho_1 - \frac{\mu}{4\pi} (\operatorname{curl} \boldsymbol{h}) \times \boldsymbol{H}_0 = \boldsymbol{0}, \quad (27.8c)$$

while the solenoidal condition (2.3) becomes

$$\operatorname{div} \boldsymbol{h} = 0. \quad (27.8d)$$

Equations (27.8*a*) to (27.8*d*) are linear in the small quantities $\boldsymbol{h}$, $\boldsymbol{v}$ and ρ_1.

Let us consider a plane wave propagating normal to the x-axis and choose the orientation of the y-axis so that the constant magnetic field vector $\boldsymbol{H}_0$, which makes an angle θ with the x-axis, lies in the (x, y)-plane. Then the components of $\boldsymbol{H}_0$ are $(H_0 \cos\theta, H_0 \sin\theta, 0)$ and, as the wave is plane and normal to the x-axis, all the dependent variables must be functions of only x and t. The Cartesian components of equations (27.8*a*) to (27.8*d*) then become †

$$\frac{\partial h_x}{\partial t} = 0, \quad (27.9a)$$

$$\frac{\partial h_y}{\partial t} - H_0 \cos\theta \frac{\partial v_y}{\partial x} + H_0 \sin\theta \frac{\partial v_x}{\partial x} = 0, \quad (27.9b)$$

$$\frac{\partial h_z}{\partial t} - H_0 \cos\theta \frac{\partial v_z}{\partial x} = 0, \quad (27.9c)$$

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial v_x}{\partial x} = 0, \quad (27.9d)$$

† The suffixes x, y, z denote the appropriate components of the vectors $\boldsymbol{h}$ and $\boldsymbol{v}$.

$$\rho_0 \frac{\partial v_x}{\partial t} + a_0^2 \frac{\partial \rho_1}{\partial x} + \frac{\mu}{4\pi} H_0 \sin\theta \frac{\partial h_y}{\partial x} = 0, \tag{27.9e}$$

$$\rho_0 \frac{\partial v_y}{\partial t} - \frac{\mu}{4\pi} H_0 \cos\theta \frac{\partial h_y}{\partial x} = 0, \tag{27.9f}$$

$$\rho_0 \frac{\partial v_z}{\partial t} - \frac{\mu}{4\pi} H_0 \cos\theta \frac{\partial h_z}{\partial x} = 0, \tag{27.9g}$$

$$\frac{\partial h_x}{\partial x} = 0. \tag{27.9h}$$

It follows immediately from equations (27.9*a*) and (27.9*h*) and the initial equilibrium state that

$$h_x = 0. \tag{27.10}$$

Combining equations (27.9*c*) and (27.9*g*) then gives

$$\frac{\partial^2 h_z}{\partial t^2} = \frac{\mu H_0^2 \cos^2\theta}{4\pi\rho_0} \frac{\partial^2 h_z}{\partial x^2} \tag{27.11}$$

and

$$\frac{\partial^2 v_z}{\partial t^2} = \frac{\mu H_0^2 \cos^2\theta}{4\pi\rho_0} \frac{\partial^2 v_z}{\partial x^2}. \tag{27.12}$$

showing that h_z and v_z propagate as Alfvén waves with Alfvén speed $b_x = \left(\frac{\mu H_0^2 \cos^2\theta}{4\pi\rho_0}\right)^{\frac{1}{2}}$ in the x-direction (cf., Example 10, § 12).

The propagation of the other dependent variables v_x, v_y, h_y and ρ_1 is less simple and is determined by equations (27.9*b*) and (27.9*d*, *e*, *f*). As solutions to linear equations describing wave motion are usually synthesised by the superposition of monochromatic harmonic waves,† let us assume that the dependent variables all have a harmonic dependence with frequency $\frac{\omega}{2\pi}$ and wavelength λ. Then for

† See Coulson, *loc. cit.*, § 10.

a plane wave moving in the positive x-direction, at any time t all dependent variables on the plane $x = \frac{\lambda\omega}{2\pi} t$ will be constant. Consequently, all the dependent variables will be functions only of $\omega t - \left(\frac{2\pi}{\lambda}\right) x$. It is customary to write

$$k = \frac{2\pi}{\lambda} \tag{27.13}$$

and to call the quantity k the **wave number.** So, as the plane waves are assumed to be harmonic, we can assume that the dependent variables all vary as $\mathscr{R} \exp\{i(\omega t - kx)\}$.

Thus, if h_y^*, v_x^*, v_y^* and ρ_1^* are the amplitudes of the harmonic disturbances of the dependent variables, we can write

$$\begin{Bmatrix} h_y \\ v_x \\ v_y \\ \rho_1 \end{Bmatrix} = \begin{Bmatrix} h_y^* \\ v_x^* \\ v_y^* \\ \rho_1^* \end{Bmatrix} \mathscr{R} \exp\{i(\omega t - kx)\}. \tag{27.14}$$

Making these substitutions in equations (27.9*b*) and (27.9*d, e, f*) we obtain a homogeneous set of equations for the amplitudes involving ω and k. The compatibility condition to be satisfied by these equations is easily seen to be

$$\begin{vmatrix} \omega & -kH_0 \sin\theta & kH_0 \cos\theta & 0 \\ 0 & -k\rho_0 & 0 & \omega \\ -\dfrac{k\mu H_0}{4\pi} \sin\theta & \omega\rho_0 & 0 & -ka_0^2 \\ \dfrac{k\mu H_0}{4\pi} \cos\theta & 0 & \omega\rho_0 & 0 \end{vmatrix} = 0. \tag{27.15}$$

A simple calculation then shows that

$$\left(\frac{\omega}{k}\right)^4 - (a_0^2+b^2)\left(\frac{\omega}{k}\right)^2 + a_0^2 b_x^2 = 0 \tag{27.16}$$

where, as before, $b = \left(\frac{\mu H_0^2}{4\pi\rho_0}\right)^{\frac{1}{2}}$ is the Alfvén speed in the direction of $\boldsymbol{H}_0$. Solving equation (27.16) for (ω/k) we obtain

$$\left(\frac{\omega}{k}\right)^2 = \tfrac{1}{2}\{(a_0^2+b^2) \pm \sqrt{(a_0^2+b^2)^2 - 4a_0^2 b_x^2}\}, \tag{27.17}$$

which is called the **dispersion relation** for the wave propagation. This relation derives its name from the fact that a medium in which the wavelength (or wave number) of a wave is frequency dependent is called a **dispersive medium**; the connection between the wavelength and the frequency being called the dispersion relation. The quantity (ω/k) is called the **phase velocity**, and is the ratio of the distance travelled by the wave for a phase change of 2π to the time taken for this phase change to occur. In general, no physical quantity travels with the phase velocity unless the medium is non-dispersive. The velocity with which a material disturbance is propagated is called the **group velocity** † which we shall denote by c. The group velocity c is related to ω and k by the relation $c = d\omega/dk$ which when applied to the dispersion relation (27.17), gives

$$\frac{d\omega}{dk} = c = \left(\tfrac{1}{2}\{(a_0^2+b^2) \pm \sqrt{(a_0^2+b^2)^2 - 4a_0^2 b_x^2}\}\right)^{\frac{1}{2}} \tag{27.18}$$

As would be expected, this is simply the result encountered previously in equations (25.31*b*, *c*) and shows that the

† See Coulson, *loc. cit.*, §§ 82 and 83.

dispersion relation (27.17) describes magnetoacoustic waves.

The construction of surfaces of normal velocity then proceeds as before while the synthesis of solutions to initial value problems by the superposition of monochromatic waves of the type just discussed can be undertaken by the usual methods.†

§ 28. Examples.

1.* Rewrite the one-dimensional isentropic flow equations for an ordinary gas in terms of the coordinates ϕ = constant and t' = constant, where $\phi(x, t) = 0$ is the wavefront trace and $t' = t$. Assuming that the density ρ and the x-component of the velocity v are analytic functions and are known along a curve S show how ρ and v can be determined in the neighbourhood of S by successively differentiating the re-written fluid equations and by using Taylor's expansion theorem. Hence show that when

$$\left(\frac{\partial \phi}{\partial t} + v\frac{\partial \phi}{\partial x}\right)^2 - a^2\left(\frac{\partial \phi}{\partial x}\right)^2 = 0,$$

the derivatives of ρ and v with respect to ϕ are indeterminate. Deduce from this that the characteristic curves ϕ = constant represent curves in the (x, t)-plane across which the normal derivatives of ρ and v become indeterminate.

2.* Let U be a column matrix with elements $u_1(x, t)$ and $u_2(x, t)$ and define the matrices $\frac{\partial U}{\partial x}$ and $\frac{\partial U}{\partial t}$ to be the column matrices formed by differentiating the elements of U with respect to x and t, respectively. By writing the one-dimensional isentropic flow equations of an ordinary gas in the form

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

† Coulson, *loc. cit.*

and

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{a^2}{\rho}\frac{\partial \rho}{\partial x} = 0,$$

show that they may be written in matrix notation as

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = 0,$$

where A is a (2×2) matrix and U is a column matrix with elements ρ and v. Change to the coordinates ϕ = constant and t' = constant, where $\phi(x, t) = 0$ defines the wavefront trace and $t' = t$, and apply the arguments used to derive the characteristic equations directly to the matrix equation obeying the usual laws for matrix manipulation. Hence show that the characteristic determinant of the fluid equations is just the characteristic determinant of A itself, namely,

$$|A - \lambda I| = 0,$$

where I is the unit matrix and $\lambda = -\left(\frac{\partial \phi}{\partial t}\right)\Big/\left(\frac{\partial \phi}{\partial x}\right)$. Show also that the characteristic equations appropriate to each of the two roots λ are given by the product of the row latent vectors of A with the matrix δU, where the δ operator is applied to each element of U.

3. Show that the characteristic determinant for the one-dimensional non-isentropic flow equations

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0,$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{a^2}{\rho}\frac{\partial \rho}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial S} = 0,$$

$$\frac{\partial S}{\partial t} + v\frac{\partial S}{\partial x} = 0,$$

with $p = p(\rho, S)$ and $\dfrac{dp}{d\rho} = a^2$, takes the form

$$\begin{vmatrix} (v-\lambda) & \rho & 0 \\ a^2/\rho & (v-\lambda) & \dfrac{1}{\rho}\left(\dfrac{\partial p}{\partial S}\right) \\ 0 & 0 & (v-\lambda) \end{vmatrix} = 0.$$

Find and interpret the three normal wave speeds λ and derive the characteristic equations appropriate to each of these.

4. Consider the three-dimensional isentropic flow equations for an ordinary gas with density ρ, sound speed a and Cartesian velocity components v_1, v_2 and v_3, respectively. Change to the coordinates ϕ = constant, y' = constant, z' = constant and t' = constant where $\phi(x, y, z, t) = 0$ is the wavefront trace in (x, y, z, t)-space and $y' = y$, $z' = z$ and $t' = t$ and show that the equations determining the jumps in ρ, v_1, v_2 and v_3 normal to $\phi = 0$ are

$$\left(\frac{\partial\phi}{\partial t} + v_1\frac{\partial\phi}{\partial x} - v_2\frac{\partial\phi}{\partial y} + v_3\frac{\partial\phi}{\partial z}\right)\left[\frac{\partial\rho}{\partial\phi}\right] + \rho\left(\frac{\partial\phi}{\partial x}\right)\left[\frac{\partial v_1}{\partial\phi}\right] + \rho\left(\frac{\partial\phi}{\partial y}\right)\left[\frac{\partial v_2}{\partial\phi}\right] + \rho\left(\frac{\partial\phi}{\partial z}\right)\left[\frac{\partial v_3}{\partial\phi}\right] = 0,$$

$$\frac{a^2}{\rho}\left(\frac{\partial\phi}{\partial x}\right)\left[\frac{\partial\rho}{\partial\phi}\right] + \left(\frac{\partial\phi}{\partial t} + v_1\frac{\partial\phi}{\partial x} + v_2\frac{\partial\phi}{\partial y} + v_3\frac{\partial\phi}{\partial z}\right)\left[\frac{\partial v_1}{\partial\phi}\right] = 0,$$

$$\frac{a^2}{\rho}\left(\frac{\partial\phi}{\partial y}\right)\left[\frac{\partial\rho}{\partial\phi}\right] + \left(\frac{\partial\phi}{\partial t} + v_1\frac{\partial\phi}{\partial x} + v_2\frac{\partial\phi}{\partial y} + v_3\frac{\partial\phi}{\partial z}\right)\left[\frac{\partial v_2}{\partial\phi}\right] = 0,$$

$$\frac{a^2}{\rho}\left(\frac{\partial\phi}{\partial z}\right)\left[\frac{\partial\rho}{\partial\phi}\right] + \left(\frac{\partial\phi}{\partial t} + v_1\frac{\partial\phi}{\partial x} + v_2\frac{\partial\phi}{\partial y} + v_3\frac{\partial\phi}{\partial z}\right)\left[\frac{\partial v_3}{\partial\phi}\right] = 0.$$

Hence show that the characteristic determinant is

$$\left(\frac{\partial\phi}{\partial t}+v_1\frac{\partial\phi}{\partial x}+v_2\frac{\partial\phi}{\partial y}+v_3\frac{\partial\phi}{\partial z}\right)^2 -a^2\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right\}=0.$$

By using the fact that in the surface $\phi = 0$ the differential $d\phi = 0$, thus giving

$$\left(\frac{\partial\phi}{\partial t}\right)dt+\left(\frac{\partial\phi}{\partial x}\right)dx+\left(\frac{\partial\phi}{\partial y}\right)dy+\left(\frac{\partial\phi}{\partial z}\right)dz=0$$

or,

$$\left(\frac{\partial\phi}{\partial t}\right)dt+(\text{grad }\phi)\,.\,d\boldsymbol{x}=0,$$

in which $d\boldsymbol{x}$ has components dx, dy and dz, show that

$$-\left(\frac{\partial\phi}{\partial t}\right)\Big/|\text{ grad }\phi\,|=\boldsymbol{n}\,.\,\boldsymbol{v}=v_n,$$

where $\boldsymbol{n} = (\text{grad }\phi)/|\text{ grad }\phi\,|$ is the spatial normal to $\phi = 0$ and v_n is the normal wavefront speed.† Hence show that the characteristic determinant can be written as

$$(v_n-\lambda)^2-a^2=0$$

with the characteristic roots $\lambda = v_n \pm a$.

5. By changing to the coordinates ϕ = constant, y' = constant, z' = constant and t' = constant, where $\phi = 0$ is the wavefront trace in (x, y, z, t)-space, $y' = y$, $z' = z$ and $t' = t$, show that the characteristic equations of the Maxwell equations

$$\text{curl }\boldsymbol{H}=\frac{4\pi\boldsymbol{j}}{c}+\frac{1}{c}\frac{\partial\boldsymbol{D}}{\partial t}$$

$$\text{curl }\boldsymbol{E}=-\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t},$$

† We have denoted the modulus of the gradient of ϕ by $|\text{ grad }\phi\,|$.

can be written in the form

$$\varepsilon\lambda\delta\boldsymbol{E}+c\boldsymbol{n}\times\delta\boldsymbol{H} = \boldsymbol{0}$$

and

$$\mu\lambda\delta\boldsymbol{H}-c\boldsymbol{n}\times\delta\boldsymbol{E} = \boldsymbol{0}$$

where $\boldsymbol{B} = \mu\boldsymbol{H}$, $\boldsymbol{D} = \varepsilon\boldsymbol{E}$, $\boldsymbol{n} = (\text{grad}\ \phi)/|\ \text{grad}\ \phi\ |$ is the unit spatial normal to $\phi = 0$ and $\lambda = -\left(\dfrac{\partial\phi}{\partial t}\right)\Big/|\ \text{grad}\ \phi\ |$ is the normal velocity of propagation of the wavefront. The symbol δ signifies either a jump in the derivative of the associated variable normal to the wavefront trace or, if the wave is entering a constant state, an infinitesimal change of the variable itself. Show, directly from the characteristic equations, that $\delta\boldsymbol{E}$, $\delta\boldsymbol{H}$ and $\boldsymbol{n}$ are mutually orthogonal and hence that electromagnetic waves are transverse waves. Find the characteristic velocities of propagation and show that the electric and magnetic disturbances both propagate with the same speed. Since the equations are linear, deduce the characteristic surfaces as functions of time.

6.* Using the equations describing the one-dimensional isentropic flow of an ordinary gas show, by forming the differentials of density ρ and velocity v along any member of the $C^{(+)}$ family of characteristics defined by $\dfrac{dx}{dt} = v+a$, that

$$\frac{d\rho}{dt} = a\frac{\partial\rho}{\partial x} - \rho\frac{\partial v}{\partial x},$$

$$\frac{dv}{dt} = a\frac{\partial v}{\partial x} - \frac{a^2}{\rho}\frac{\partial\rho}{\partial x},$$

along any $C^{(+)}$ characteristic. Hence show that these equations may be integrated to give

$$v + \int\frac{a\,d\rho}{\rho} = \text{constant along any } C^{(+)} \text{ characteristic.}$$

Show also that

$$v - \int \frac{a\,d\rho}{\rho} = \text{constant along any } C^{(-)} \text{ characteristics,}$$

where the $C^{(-)}$ characteristics are determined by $\dfrac{dx}{dt} = v - a$.

If the $C^{(-)}$ characteristics are parameterised by requiring that $\phi(x, 0) = \alpha(x)$ and the $C^{(-)}$ characteristics are parameterised by requiring that $\phi(x, 0) = \beta(x)$, show that

$$v + \int \frac{a\,d\rho}{\rho} = r_{+}(\beta) \text{ along } C^{(+)} \text{ characteristics}$$

and

$$v - \int \frac{a\,d\rho}{\rho} = r_{-}(\alpha) \text{ along } C^{(-)} \text{ characteristics,}$$

where the $r_{+}(\alpha)$ and $r_{-}(\beta)$ are functions only of α and β, respectively, and are called **Riemann invariants.**

7.* Suggest how the Riemann invariants of Example 6 may be used to determine ρ and v at a point P in the (x, t)-plane when the $C^{(+)}$ and $C^{(-)}$ characteristics through P are known and intersect an initial curve S in the (x, t)-plane along which ρ and v have been specified. Show that if a $C^{(+)}$ characteristic $C_0^{(+)}$ is adjacent to a constant state (ρ_0, v_0) then $r_{-}(\alpha) = r_{-0}$ is constant in the flow adjacent to the constant state, which is then called **simple wave** flow, and that in the region of simple wave flow the $C^{(+)}$ family of characteristics becomes a family of straight lines.†

8. Derive the characteristic determinant for the Lundquist equations and show that the characteristic relation is

$$c_n(c_n^2 - b_x^2)\{(c_n^2 - a^2)(c_n^2 - b_x^2) - c_n^2(b^2 - b_x^2)\} = 0,$$

† Compare with, Rutherford, *Fluid Dynamics*, 1959, § 49. This type of flow also includes the two-dimensional expansion process experienced by an ordinary gas when expanding round a corner. This is often called a **Prandtl-Meyer** expansion.

where a is the speed of sound, $b = \sqrt{\dfrac{\mu H^2}{4\pi\rho}}$ is the Alfvén speed, $b_x = \sqrt{\dfrac{\mu H_x^2}{4\pi\rho}}$ is the Alfvén speed in the x-direction and $c_n = |\lambda - v_n|$ is the speed of the wavefront relative to the fluid.

9. Show by considering the Lundquist characteristic equations that the disturbance characterising an entropy wave in which $H_x = 0$ takes the form

$$\delta \boldsymbol{H} = \varepsilon \boldsymbol{K},$$

$$\delta \boldsymbol{v} = \varepsilon \boldsymbol{t},$$

$$\delta S = \varepsilon_1,$$

and

$$\delta\rho = -\frac{1}{a^2}\left(\frac{\partial p}{\partial S}\varepsilon_1 + \frac{\mu}{4\pi}\boldsymbol{H}.\boldsymbol{K}\varepsilon\right),$$

where ε and ε_1 are parameters describing the size of the infinitesimal jumps, $\boldsymbol{t}$ is a unit vector perpendicular to $\boldsymbol{n}$, and $\boldsymbol{K}$ is an arbitrary vector perpendicular to $\boldsymbol{n}$. Hence show that the transverse components of velocity and magnetic field may undergo an arbitrary infinitesimal jump but that the density and pressure jumps are subject to the single condition

$$\delta p^* = \delta\left(p + \frac{\mu \boldsymbol{H}^2}{8\pi}\right) = 0.$$

10. By considering the magnetohydrodynamic characteristic equations appropriate to an incompressible fluid show that in an Alfvén wave

$$c_n = b_x,$$

$$\delta \boldsymbol{v}_t = \mp\sqrt{\frac{\mu}{4\pi\rho}}\,\mathrm{sgn}\,(H_x)\delta \boldsymbol{H}_t,$$

$$\delta p = -\left(\frac{\mu}{4\pi}\right) H_y \delta H_y$$

and

$$\delta p^* = 0,$$

where the suffix t denotes a transverse vector with respect to the wavefront normal. Compare the Alfvén wave with a general transverse wave.

11. Assume a constant state with zero flow velocity v_x and by considering the propagation of a weak plane wave from a plane source which is itself distant R_0 from the origin with its normal inclined at an angle θ to the x-axis, prove that the wavefront emanating from a cylindrical source of radius R_0 that is centred on the origin is given by

$$x = R_0 \cos\theta \pm \left(c_n \cos\theta - \frac{dc_n}{d\theta}\sin\theta\right) t,$$

$$y = R_0 \sin\theta \pm \left(c_n \sin\theta + \frac{dc_n}{d\theta}\cos\theta\right) t.$$

Show that the equation of the slow wavefront is given by

$$\frac{x}{b} = \frac{R_0 \cos\theta}{b} \pm \cos\theta \left\{\frac{c_s}{b} + \frac{s \sin^2\theta}{(c_s/b)[(1+s)^2 - 4s\cos^2\theta]^{\frac{1}{2}}}\right\} t$$

and

$$\frac{y}{b} = \frac{R_0 \sin\theta}{b} \pm \sin\theta \left\{\frac{c_s}{b} - \frac{s \cos^2\theta}{(c_s/b)[(1+s)^2 + 4s\cos^2\theta]^{\frac{1}{2}}}\right\} t.$$

Derive the corresponding equations for the wavefronts of the transverse and fast waves.

12. Assume one-dimensional isentropic magnetohydrodynamic flow in which $H_x \equiv 0$, and show that v_y and v_z are constant along lines of flow. Hence prove that

$$\frac{D}{Dt}\log\left(\frac{H_y}{\rho}\right) = 0, \quad \frac{D}{Dt}\log\left(\frac{H_z}{H_y}\right) = 0,$$

and that

$$H_y = k_1 \rho, \quad H_z = k_2 H_y,$$

where k_1 and k_2 are constants. Combine the remaining equations to show that for a polytropic gas

$$\frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x} + \rho \frac{\partial v_x}{\partial x} = 0,$$

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + \left(\frac{\mu k_1^2}{4\pi} + \frac{a^2}{\rho}\right) \frac{\partial \rho}{\partial \rho} = 0.$$

Prove that these equations are hyperbolic and that c_n, the wave speed relative to the fluid, is given by

$$c_n = \pm (a^2 + b_y^2)^{\frac{1}{2}},$$

which is only density dependent.

13. Sketch the geometrical construction of the slow wavefront from the surface of slow normal velocity that is appropriate to a fixed point source in a uniform region. Taking the x-axis parallel to the magnetic field $\boldsymbol{H}$, and letting θ denote the angle between $\boldsymbol{H}$ and the wavefront normal, show that the cusp of the slow wavefront that lies on the positive x-axis is generated as the limit of the envelope of the surface of slow normal velocity as $\theta \to \frac{1}{2}\pi$. By considering the limiting form as $\theta \to \frac{1}{2}\pi$ of the parametric representation of x as a function of θ for the slow wavefront, slow that at time t the slow wavefront cusp on the positive x-axis is distant from the point source by an amount $abt/(a^2+b^2)^{\frac{1}{2}}$, where a is the sound velocity and b is the Alfvén speed.

CHAPTER V

MAGNETOHYDRODYNAMIC SIMPLE WAVES

§ 29. **One-dimensional wave propagation.** Of the many possible types of magnetohydrodynamic wave motion that may occur, just as in ordinary gas dynamics,† the particularly simple wave motions belonging to the class of **one-dimensional** flows that can exist **adjacent to a region of constant state** are of fundamental importance. The importance of these flows stems from the fact that they describe the basic transition flows of magnetohydrodynamics such as the compression and expansion processes that must take place if a gas is to undergo a change from one constant state to another. Here we use the expression " region of constant state " to signify a region in which ρ, v_x, H_x, ..., S all have constant values. Later we shall show how these special flows may be combined with certain non-differentiable solutions of the magnetohydrodynamic equations called shocks in order to solve some simple but interesting magnetohydrodynamic flow problems.

So, taking equations (25.18) to (25.24) as our starting point, let us first notice that since, by supposition, the wave is adjacent to a region of constant state, we may use the results of § 24 to replace the jumps $\delta\rho$, δv_x, ..., δS in the derivatives of ρ, v_x, ..., δS by the differentials $d\rho$, dv_x, ..., dS. By so doing we obtain the following **one-dimensional** form

† See Rutherford, *Fluid Dynamics*, 1959, §§ 49 and 50.

of the **magnetohydrodynamic characteristic equations**

$$\mp c_n d\rho + \rho dv_x = 0, \tag{29.1}$$

$$\mp c_n \rho dv_x + a^2 d\rho + \left(\frac{\partial p}{\partial S}\right) dS + \frac{\mu}{4\pi} H_y dH_y = 0, \tag{29.2}$$

$$\mp c_n \rho dv_y - \frac{\mu}{4\pi} H_x dH_y = 0, \tag{29.3}$$

$$\mp c_n \rho dv_z - \frac{\mu}{4\pi} H_x dH_z = 0, \tag{29.4}$$

$$\mp c_n dH_y + H_y dv_x - H_x dv_y = 0, \tag{29.5}$$

$$\mp c_n dH_z - H_x dv_z = 0, \tag{29.6}$$

and

$$\mp c_n dS = 0, \tag{29.7}$$

while the solenoidal condition div $\boldsymbol{H} = 0$ shows that

$$H_x = H_{0x} \tag{29.8}$$

is constant.

Then, by successively identifying c_n with the roots c_f, c_s, b_x and 0 obtained in § 25, we can define special one-dimensional fast, slow, transverse and entropy waves. These waves represent particularly simple solutions of the magnetohydrodynamic characteristic equations and, although general wave motion is governed by the Lundquist partial differential equations, we shall now show that in this special case the behaviour of all the physical quantities is in fact determined by a single ordinary differential equation. We shall also show that it is a direct consequence of this result that the dependent variables ρ, v_x, H_x, ..., S which are involved in the wave motion adjacent to the constant state may all be expressed as functions of one of these dependent variables, say ρ. Magnetohydrodynamic waves having this very special property will be called **magnetohydrodynamic simple waves.**†

§ 30. Contact surfaces and transverse simple waves. Let us start by discussing contact surfaces and transverse simple waves since, unlike fast and slow simple waves, certain of the dependent variables involved in them can undergo finite jumps across the wavefront. This property simplifies their description by allowing the use of jump conditions in place of differential equations. The so-called **entropy wave** or **contact surface** is of particular value when matching flows involving dissimilar fluid states, while transverse simple waves assist in aligning the magnetic fields in adjoining flows.

In § 26(*b*, *c*) we have already examined the relationships that exist between the jumps in the derivatives of the dependent variables, it being assumed there that the functions themselves were continuous. However, we are now dealing with differentials of the functions and so, under certain conditions, finite jumps in the dependent variables themselves may occur. In particular, the fact that in transverse wave motion the jumps δv_z and δH_z were possible now implies that finite jumps must be allowed for in the z-direction. Thus, the device used to simplify the equations in § 25, whereby the axes were rotated to make $H_z = 0$, is no longer of value since H_z might now experience a finite jump across the wavefront.

To allow for this possibility we must reconsider equations (25.8) to (25.16). When deriving the characteristic equations (25.18) to (25.24) we omitted the terms involving H_z in equations (25.9) and (25.14), otherwise our results were quite general. If, now, we retain these terms, equations (25.19) and (25.23) must be supplemented by the addition

† It is possible to re-phase this definition into a mathematical statement concerning a singular flow for which the hodograph transformation (Rutherford, *loc. cit.*, §§ 38 and 52) becomes invalid.

of the terms $\frac{\mu}{4\pi} H_z \delta H_z$ and $H_z \delta v_x$, respectively. Thus, for the consideration of finite jumps across contact surfaces and transverse simple waves we must supplement equations (29.2) and (29.6) by the addition of the terms $\frac{\mu}{4\pi} H_z dH_z$ and $H_z dv_x$, respectively.

A contact surface across which finite jumps can occur is then described by the equations (29.1) to (29.8) modified with $c_n = 0$. Equation (29.7) shows that dS is arbitrary and hence that the entropy can experience an arbitrary finite jump across a contact surface. The remaining modified equations show that provided $H_x \neq 0$, $\boldsymbol{H}$, $\boldsymbol{v}$ and p are continuous across a contact surface and that there is no mass flow across the wavefront. The density and entropy are related to the pressure by the equation of state of a polytropic gas (25.6) and by the expression

$$a^2 d\rho + \left(\frac{\partial p}{\partial S}\right) dS = 0 \tag{30.1}$$

derived from the modified equation (29.2).

The fact that the velocity vector $\boldsymbol{v}$ must be continuous across a magnetohydrodynamic contact surface gives rise to an important difference between classical fluid dynamics and magnetohydrodynamics. In ordinary fluid dynamic **shear flow**, in which adjacent fluids are in contact but do not mix, the only requirement to be satisfied across their interface is that the normal components of fluid velocity on either side of the interface should always be equal. Thus for a shear flow in an ordinary fluid a tangential velocity discontinuity across the interface is permitted, whereas in magnetohydrodynamic flow in which $H_x \neq 0$ it is not. A magnetohydrodynamic shear flow is only possible if $H_x = 0$, and then a discontinuity can also occur in the transverse magnetic field $\boldsymbol{H}_t$ (see Example 1, § 37).

Integrating the one-dimensional form of equation (2.9) with respect to x and differencing the result across the wavefront shows that a transverse surface current flows in the wavefront whenever $[\boldsymbol{H}_t] \neq 0$. The transverse surface current **J** is easily seen to be given by

$$\boldsymbol{J} = \frac{c}{4\pi}\, \boldsymbol{i} \times [\boldsymbol{H}_t], \tag{30.2}$$

where $\boldsymbol{i}$ is the unit vector along the x-axis.

A particular limiting case of a contact surface occurs when the flow takes place adjacent to a vacuum region. We have already seen that when $H_x \neq 0$ the pressure p will be continuous across the contact surface. Combining this result with the equation of state for a polytropic gas (25.6) and the fact that the density is zero in the vacuum region then shows that the density is also zero behind the contact surface. Hence, since this implies that $d\rho = 0$, equation (30.1) then shows that $dS = 0$ across such a degenerate contact surface. In the next Section we shall see that a contact surface of this type occurs as a limiting form of a magnetohydrodynamic simple wave and serves to separate it from a vacuum field.

If, alternatively, we assume that $H_x = 0$, equations (29.3) and (29.4) show that the transverse magnetic field vector $\boldsymbol{H}_t$ can experience a discontinuity across the contact surface. It then follows directly from the modified equation (29.2) that, even though the fluid density is zero in the vacuum region ahead of the contact surface, it can be non-zero behind it. This type of contact surface is also related to a limiting form of magnetoacoustic wave and separates it from a vacuum field.

The modified equations (29.1) to (29.7) describe finite **transverse simple waves** when we set $c_n = b_x$, the Alfvén speed in the x-direction. It is easily shown that

$$d\rho = dv_x = dS = 0, \tag{30.3}$$

while $d\boldsymbol{H}_t$ is arbitrary in size and is related to $d\boldsymbol{v}_t$ by the expression

$$d\boldsymbol{v}_t = \mp \sqrt{\frac{\mu}{4\pi\rho}}\,\text{sgn}\,(H_x)d\boldsymbol{H}_t. \qquad (30.4)$$

Using results (30.3) in the modified equation (29.2) gives

$$H_y dH_y + H_z dH_z = 0 \qquad (30.5)$$

or

$$d(\boldsymbol{H}_t^2) = 0, \qquad (30.6)$$

showing that although $\boldsymbol{H}_t$ experiences an arbitrary jump across the transverse wavefront, the magnitude of $\boldsymbol{H}_t$ remains constant.

The solution $\boldsymbol{H}_t$ to these equations may be parameterised in terms of an angle Ψ and expressed in the form

$$H_y = H_{0t} \sin \Psi, \quad H_z = H_{0t} \cos \Psi, \qquad (30.7)$$

where H_{0t} is the magnitude of $\boldsymbol{H}_t$. Hence the transverse magnetic field vector $\boldsymbol{H}_t$ can experience a finite rotation when crossing a transverse simple wave and results (30.3) show that ρ, v_x and S will remain constant. Transverse simple waves thus describe **rotational discontinuities** in the transverse magnetic field. It is easy to see that the fluid pressure p and the total pressure $p^* = p + \dfrac{\mu \boldsymbol{H}^2}{8\pi}$ will be continuous across a transverse simple wave.

§ 31. Fast and slow simple waves. Fast and slow magnetohydrodynamic simple waves propagate with the normal wave speeds c_f and c_s, respectively. The discussion of § 26(*a*), in which H_z was set equal to zero, showed that no *z*-components of velocity or magnetic field are involved in the compatible jump conditions for the derivatives. Accordingly, when we consider finite fast and slow simple wave disturbances, we may assume that $H_z = 0$. Thus fast and slow magnetohydrodynamic simple waves, or **fast and slow**

magnetoacoustic simple waves, are described by equations (29.1) to (29.8) in which c_n is identified with c_f or c_s, respectively.

Since we shall assume that c_n is neither equal to b_x nor to zero, equation (29.7) then shows that S is constant. Combining equations (29.4) and (29.6) now gives

$$(c_n^2 - b_x^2)dH_z = 0$$

or, because $c_n \neq b_x$,

$$dH_z = 0. \tag{31.1}$$

Hence, in the special case of fast and slow magnetoacoustic simple waves, the flow is completely described by the four equations (29.1) to (29.3) and (29.5). These have as their characteristic determinant the expression

$$\begin{vmatrix} \mp c_n & \rho & 0 & 0 \\ a^2 & \mp c_n\rho & 0 & \dfrac{\mu H_y}{4\pi} \\ 0 & 0 & \mp c_n & -\dfrac{\mu H_x}{4\pi\rho} \\ 0 & H_y & -H_x & \mp c_n \end{vmatrix} = 0, \tag{31.2}$$

which we now re-write as

$$c_n^2 b_y^2 = (c_n^2 - a^2)(c_n^2 - b_x^2). \tag{31.3}$$

Comparing the roots of this equation with equations (25.31*b*, *c*) we see, as we would expect, that c_n may take the values c_f or c_s.

By eliminating dv_x between equations (29.1) and (29.2) we find that

$$(a^2 - c_n^2)d\rho + \frac{\mu}{8\pi} d(H_y^2) = 0, \tag{31.4}$$

while by introducing the non-dimensional variables α and β through the equations

$$\alpha = \frac{c_n^2}{a^2} \tag{31.5}$$

and

$$\beta = \frac{a^2}{b_x^2}, \tag{31.6}$$

equation (31.3) becomes

$$H_y^2 = (\alpha-1)(\beta-\alpha^{-1})H_x^2. \tag{31.7}$$

Then, combining equations (31.4) and (31.7), dividing by ρa^2 and using the definition of β we have, since H_x = constant, that

$$(1-\alpha)\frac{d\rho}{\rho}+\frac{1}{2\beta}d[(\alpha-1)(\beta-\alpha^{-1})] = 0. \tag{31.8}$$

However, β may be written in the form

$$\beta = \frac{a^2}{\mu H_x^2/4\pi\rho}$$

which, for a polytropic gas with pressure p and adiabatic exponent γ, becomes

$$\beta = \frac{\gamma p}{\mu H_x^2/4\pi}.$$

Now, since H_x = constant, β is proportional to p and so we can write

$$p = \hat{p}\beta \tag{31.9}$$

and

$$\rho = \hat{\rho}\beta^{1/\gamma}, \tag{31.10}$$

where the constants $\hat{p}$ and $\hat{\rho}$ are given by

$$\hat{p} = \frac{1}{\gamma\rho b_x^2} \text{ and } \hat{\rho} = \left(\frac{p}{A(S)}\right)^{1/\gamma}. \tag{31.11}$$

By defining $\hat{a}^2$ by the relation

$$\hat{a}^2 = \gamma\hat{p}/\hat{\rho} \tag{31.12}$$

it then follows directly from the definition of α that

$$c_n^2 = \hat{a}^2\alpha\beta^{(1-1/\gamma)}. \tag{31.13}$$

If we now make use of the relationship between β and ρ that is displayed in equation (31.10) we can re-write equation (31.8) as the ordinary differential equation

$$\frac{d\beta}{d\alpha} = \left(\frac{\gamma}{2-\gamma}\right)\frac{(\alpha^2\beta-1)}{\alpha^2(\alpha-1)}. \tag{31.14}$$

This equation, which was first derived by K. O. Friedrichs, determines the variation of β as a function of α throughout the simple wave region. Later, we shall use it in conjunction with other equations from this section in order to derive the behaviour of all physical quantities across simple waves.

Since $\alpha\beta = c_n^2/b_x^2$, it follows directly from inequality (26.2*a*) that for the fast wave region, denoted by $(+)$,

$$\alpha_+\beta \geqq 1 \quad \text{and} \quad \alpha_+ \geqq 1, \tag{31.15a}$$

while for the slow wave region, denoted by $(-)$, it follows from inequality (26.2*b*) that

$$\alpha_-\beta \leqq 1 \quad \text{and} \quad \alpha_- \leqq 1, \tag{31.15b}$$

with $\beta \geqq 0$ and $\alpha_- \geqq 0$. Hence the slow wave region lies to the left of the line $\alpha = 1$ and the curve $\beta = 1/\alpha$, while the fast wave region lies to the right of these lines. Equation (31.7) show that H_y is complex outside the $(+)$ and $(-)$ regions which are illustrated in Fig. 16.

§ 32. The singularities of the equation connecting α and β. An examination of equation (31.14) shows that $d\beta/d\alpha$ becomes infinite along the lines $\alpha = 0$ and $\alpha = 1$. However, the point P on the line $\alpha = 1$, at which $\beta = 1$, is exceptional, for there both the numerator and the denominator of the right-hand side of equation (31.14) vanish.

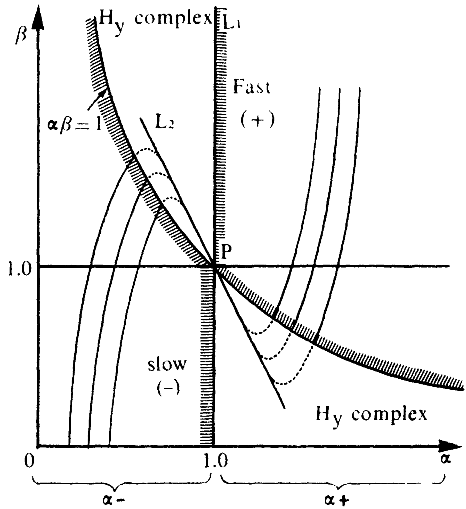

FIG. 16

To determine the behaviour of the solution in the neighbourhood of P we set

$$\tilde{\alpha} = 1-\alpha \quad \text{and} \quad \tilde{\beta} = 1-\beta \tag{32.1}$$

when, to first order, equation (31.14) becomes

$$\frac{d\tilde{\beta}}{d\tilde{\alpha}} = \left(\frac{\gamma}{2-\gamma}\right)\left(\frac{2\tilde{\alpha}+\tilde{\beta}}{\tilde{\alpha}}\right). \tag{32.2}$$

It is convenient to parameterise equation (32.2) in terms of a single valued differentiable parameter s by writing

$$\frac{d\tilde{\beta}}{ds} = \left(\frac{\gamma}{2-\gamma}\right)(2\tilde{\alpha}+\tilde{\beta}) \tag{32.3a}$$

and

$$\frac{d\tilde{\alpha}}{ds} = \tilde{\alpha}. \tag{32.3b}$$

Thus the integral curves of equation (31.14) in the neighbourhood of point P are determined by equations (32.3*a*, *b*). These equations are linear and so we may seek a solution in the form

$$\tilde{\alpha} = Ae^{\lambda s}, \quad \tilde{\beta} = Be^{\lambda s}. \tag{32.4}$$

Substituting these values into equations (32.3*a*, *b*) we obtain the following equations connecting A, B and λ:

$$2\gamma^* A + (\gamma^* - \lambda)B = 0, \tag{32.5a}$$

$$(\lambda - 1)A = 0, \tag{32.5b}$$

where

$$\gamma^* = \left(\frac{\gamma}{2-\gamma}\right). \tag{32.5c}$$

If equations (32.5*a*, *b*) are to be compatible, the determinant of the coefficients of A and B must vanish from which we find that either $\lambda = 1$ or $\lambda = \gamma^*$.

Let us now set $\lambda_1 = \gamma^*$ and $\lambda_2 = 1$ and let $A = A_i$ and $B = B_i$ be non-trivial solutions of the homogeneous equations (32.5*a*, *b*) corresponding to $\lambda = \lambda_i (i = 1, 2)$ respectively. We then find that

$$A_1 = 0, \; A_2 = 1, \; B_1 = 1, \; B_2 = \frac{-2\gamma^*}{(\gamma^* - 1)}. \tag{32.6}$$

Thus for $i = 1$ we have

$$\tilde{\alpha} = 0, \quad \tilde{\beta} = e^{\gamma^* s}, \tag{32.7a}$$

while for $i = 2$,

$$\tilde{\alpha} = e^s, \; \tilde{\beta} = \frac{-2\gamma^*}{(\gamma^* - 1)} e^s. \tag{32.7b}$$

The general solution to equations (32.3*a*, *b*) which is a linear combination of these results thus has the parametric form

$$\tilde{\alpha} = c_2 e^s \tag{32.8a}$$

and

$$\tilde{\beta} = c_1 e^{\gamma^* s} - c_2 \left(\frac{2\gamma^*}{\gamma^* - 1}\right) e^s,$$

where c_1 and c_2 are arbitrary constants.

Now, apart from this general solution, equations (32.7*a*, *b*) show that two special straight line solutions L_1 and L_2 are defined by the equations

$$L_1 \colon \; \tilde{\alpha} = 0 \tag{32.9a}$$

and

$$L_2 \colon \; \tilde{\beta} = \frac{-2\gamma^*}{(\gamma^* - 1)} \tilde{\alpha}. \tag{32.9b}$$

All solutions of equation (31.14) are tangent to the line L_2 at the point P which is called a **node** † of the equation. Consequently, the common gradient of all the integral curves at the point P is just the gradient of the line L_2, and so at point P

$$\frac{d\beta}{d\alpha} = -\left(\frac{2\gamma^*}{\gamma^* - 1}\right) = \frac{-\gamma}{\gamma - 1}. \tag{32.10}$$

The lines L_1 and L_2 are shown together with representative integral curves in Fig. 16 where, since $\gamma > 1$, the line L_2 has a negative gradient. Lines L_1 and L_2 are coincident for $\gamma = 1$. Thus, as γ increases, so L_2 tends asymptotically to the tangent to the curve $\beta = 1/\alpha$ at point P. Inspection of Fig. 16 together with inequalities (31.14*a*, *b*) shows that fast waves occur in the region marked (+) and slow waves occur in the region marked (−).

In the slow wave region (−) we have $\alpha_- \leqq 1$, $\beta\alpha_- \leqq 1$ and so $\beta\alpha_-^2 \leqq \beta\alpha_- \leqq 1$, while $-1 \leqq \alpha_- - 1 \leqq 0$. Thus it

† See, for example, Ince, *Integration of Ordinary Differential Equations*, 1952, p. 37.

follows at once from equation (31.14) that $\frac{d\beta}{d\alpha}$ is a positive function of α_- throughout the slow wave region. Using this result to examine the behaviour of $\frac{d^2\beta}{d\alpha^2}$ shows that $d\beta/d\alpha$ is a positive monotonic decreasing function of α_- throughout the slow wave region $(-)$. Similarly, $d\beta/d\alpha$ is a positive monotonic increasing function of α_+ throughout the fast wave region $(+)$.

It follows directly from this analysis of the node at point P that no physical process exists by which fast and slow waves may be connected. To see this we notice that the only way to connect a solution in the slow wave region with one in the fast wave region is through the point P. However, as we have already shown, apart from the special solution corresponding to the line L_1, all the other solutions are tangent to the line L_2 at P having a negative slope $d\beta/d\alpha = -\gamma/(\gamma-1)$ at that point.

Consequently, since $d\beta/d\alpha$ has been shown to be positive throughout the fast and slow wave regions, the general fast and slow wave solutions cannot be joined through P. However, there still remains the line L_1 linking the fast and slow regions. The equation of the line L_1 is $\alpha = 1$ and so, using the definition of α, we see that everywhere along L_1 we must have $c_n^2 = a^2$. Consequently the disturbance corresponding to line L_1 is only an ordinary sound wave and we have thus demonstrated the impossibility of connecting general slow and fast waves.

§ 33. Generalised Riemann invariants. Examples 6 and 7 of § 28 have already indicated something of the properties and use of the ordinary Riemann invariants which can be defined for pairs of hyperbolic first-order equations involving two dependent and two independent variables. Specifically, they indicated how in an ordinary two-dimensional isentropic gas flow an invariant functional relationship exists along each characteristic curve between the density and the velocity. This invariant expression, called a **Riemann invariant**, has a different (constant) value along each characteristic and so, if the constant values associated with the $C^{(+)}$ and $C^{(-)}$ characteristics through any point P are known, the invariant relations may be solved to determine the density and the velocity at point P. In the special case of simple waves in an ordinary gas (see Example 7, § 28), one of the two families of characteristics reduces to a family of straight lines, thus simplifying the problem and enabling the flow to be completely determined in the simple wave region.

Since the equations of one-dimensional magnetohydrodynamic flow (29.1) to (29.8) involve more than two dependent variables, these ordinary Riemann invariants are not directly applicable. However, we shall now show how equation (31.14) can be used to introduce generalised Riemann invariants in magnetohydrodynamic one-dimensional simple wave flow. These generalised Riemann invariants are no longer constant along characteristics as is the case with ordinary Riemann invariants. Nevertheless one set of characteristics is still a family of straight lines along each of which all the dependent variables can be expressed in terms of one dependent variable, say the density ρ (this property is also true in ordinary simple waves).

Let us start by noticing that equation (31.14) has an integrating factor $(\alpha-1)^{-\gamma/(2-\gamma)}$. Using this to integrate equation (31.14) we obtain the following expression for the generalised Riemann invariants K_+ and K_- appropriate to the slow wave region $(-)$ and the fast wave region $(+)$, respectively:

$$K_\pm = \beta \mid \alpha_\pm - 1 \mid^{-\gamma^*} \pm \gamma^* \int_{\alpha_0}^{\alpha} \alpha_\mp^{-2} \mid \alpha_\pm - 1 \mid^{-(1+\gamma^*)} d\alpha_\pm, \quad (33.1)$$

where α_0 is the value of α at the start of the simple wave, and the upper and lower $\pm$ signs correspond to the fast and slow waves, respectively. The generalised Riemann invariants $K_\pm$ depend only on α and β, which are in turn dependent on the magnetic field and the density. For this reason the invariants $K_\pm$ are often called **magnetic Riemann invariants.**

To obtain the appropriate x-invariants involving the velocity v_x we use the fact that, in terms of α and β, equation (29.1) can be re-written in the form

$$dv_x = \varepsilon\gamma^{-1}\hat{a}\alpha_\pm^{\frac{1}{2}}\beta^{-\frac{1}{2}(1+(1/\gamma))}d\beta, \tag{33.2}$$

where $\varepsilon = +1$ or -1, corresponding to the minus or plus signs of c_n in equation (29.1), respectively. Integrating this equation then gives the generalised x-Riemann invariants K_{x+} and K_{x-} appropriate to the fast wave region $(+)$ and the slow wave region $(-)$, respectively, where

$$K_{x\pm} = v_x - \varepsilon\gamma^{-1}\int_{\beta_0}^{\beta} \hat{a}\alpha_\pm^{\frac{1}{2}}\beta^{-\frac{1}{2}(1+(1/\gamma))}d\beta. \tag{33.3}$$

Here β_0 is the value of β at the start of the simple wave, ε has the same meaning as in equation (33.2) and the $\pm$ signs correspond.

By eliminating dH_y from equations (29.3) and (29.5) and using equations (31.5) to (31.7) and (33.2), it follows that (see Example 3, § 37)

$$dv_y = \mp\varepsilon\gamma^{-1}\hat{a}\beta^{-\frac{1}{2}(1+(1/\gamma))}\sqrt{\frac{\alpha_\pm - 1}{\alpha_\pm\beta - 1}}\ \mathrm{sgn}\,(H_{y0}H_x)d\beta, \tag{33.4}$$

where ε has the same meaning as before, H_{y0} denotes the value of H_y in the constant state ahead of the wave, $\mathrm{sgn}\,(H_{y0}H_x)$ denotes the sign of the expression $H_{y0}H_x$ and the upper and lower signs correspond.

Integration of this equation then yields the generalised y-Riemann invariants K_{y+} and K_{y-} appropriate to the fast wave region $(+)$ and the slow wave region $(-)$, respectively, where

$$K_{y\pm} = v_y \pm \varepsilon\gamma^{-1} \operatorname{sgn}(H_{y0}H_x) \int_{\beta_0}^{\beta} \hat{a}\beta^{-\frac{1}{2}(1+(1/\gamma))} \sqrt{\frac{\alpha_\pm - 1}{\alpha_\pm \beta - 1}}\, d\beta. \tag{33.5}$$

The upper and lower $\pm$ signs again correspond and ε and β_0 have their previous meanings.

Since we are assuming that the gas is polytropic, we also have the fourth invariant G given by the expression

$$G = p\rho^{-\gamma}, \tag{33.6}$$

which is implied by the equation of state (25.6) and by the fact that S is constant.

The lower limits α_0 and β_0 of the integrals determining the invariants $K_\pm$, $K_{x\pm}$ and $K_{y\pm}$ all correspond to the initial conditions that are present at the start of the simple wave. These are related to $H_x(= H_{x0})$ and to H_{y0} by equation (31.7). Solving for α_0 we find that

$$\alpha_{0\pm} = \frac{(H_{y0}^2+(1+\beta_0)H_{x0}^2) \pm \sqrt{(H_{y0}^2+(1+\beta_0)H_{x0}^2)^2 - 4\beta_0 H_{x0}^4}}{2\beta_0 H_{x0}^2}, \tag{33.7}$$

where the upper and lower signs correspond, respectively. Equation (33.7) thus determines the two initial values $\alpha = \alpha_{0+}$ and $\alpha = \alpha_{0-}$ that correspond, respectively, to the fast wave (+) and the slow wave (−).

The initial conditions (α_0, β_0) determine the constants $K_\pm$ associated with the magnetic Riemann invariants through the relations

$$K_\pm = \beta_0 \left| \alpha_{0\pm} - 1 \right|^{\gamma^*}. \tag{33.8}$$

The functional dependence of β on α, as determined by equation (33.1), is of fundamental importance in the determination of the variation of physical quantities across magnetoacoustic simple waves. Although it is only possible to evaluate the integral in equation (33.1) analytically when γ^* assumes special values (e.g., in the case of a monatomic gas, since then $\gamma = 5/3$ and $\gamma^* = 5$), a numerical solution by quadratures is always possible.† Alternatively, equation (31.14) can be numerically integrated using a technique such as the Runge-Kutta method.‡

The variation of the pressure p, the density ρ, the magnetic field H_y and the x and y-components of velocity v_x and v_y, respectively, are then determined by the equations:

$$p = \hat{p}\beta, \tag{31.9}$$

$$\rho = \hat{\rho}\beta^{1/\gamma}, \tag{31.10}$$

$$H_y^2 = (\alpha - 1)(\beta - \alpha^{-1})H_x^2, \tag{31.7}$$

$$K_{x\pm} = v_x - \varepsilon\gamma^{-1}\int_{\beta_0}^{\beta} \hat{a}\alpha_{\pm}^{\frac{1}{2}}\beta^{-\frac{1}{2}(1+(1/\gamma))}d\beta \tag{33.3}$$

and

$$K_{y\pm} = v_y \pm \varepsilon\gamma^{-1}\int_{\beta_0}^{\beta} \hat{a}\beta^{-\frac{1}{2}(1+(1/\gamma))}\sqrt{\frac{\alpha_{\pm}-1}{\alpha_{\pm}\beta - 1}}\,d\beta. \tag{33.5}$$

§ 34. The variation of physical quantities across fast and slow waves. A qualitative knowledge of the behaviour of physical quantities across magnetoacoustic simple waves can easily be obtained from the results of the two previous sections. If we consider a wave emanating from a given initial state characterised by the point (α_0, β_0), then fast waves will result if (α_0, β_0) lies in the $(+)$ region of the (α, β)-plane, whereas slow waves will result if (α_0, β_0) lies in the $(-)$ region of the plane. The unique integral curve passing through (α_0, β_0) will then determine, through its particular functional dependence of β on α, the variation of all the physical quantities involved in the flow.

† See Noble, *Numerical Methods 2: Differences, Integration and Differential Equations*, 1964, Chapter IX.
‡ Noble, *loc. cit.*, § 10.5.

Thus, it follows directly from equations (31.9), (31.10) and (33.2) that when $\varepsilon = 1$,

v_x changes in the same sense as p, ρ and β,

while when $\varepsilon = -1$,

v_x changes in the opposite sense to p, ρ and β.

A **compression wave** is thus characterised by increasing β, while an **expansion** or **rarefaction wave** is characterised by decreasing β. The constant ε is $+1$ for a wave advancing in the positive x-direction and -1 for a wave advancing in the negative x-direction (cf. Equations (25.32*a*, *b*)). The behaviour of v_x, p and ρ across compression and rarefaction magnetoacoustic simple waves is thus directly analogous to their behaviour in the corresponding wave phenomena in an ordinary gas.

Let us now consider a slow wave propagating in the positive x-direction into a constant state in which $p = p_0$, $\rho = \rho_0$, $v_x = v_y = 0$, $H_x > 0$ and $H_{y0} > 0$. This constant state will be characterised by a point (α_0, β_0) in the $(-)$ region of the (α, β)-plane.

If we assume a slow compression wave starting from the point (α_0, β_0) then β, and hence α_-, increase. Equation (31.7) then shows that H_y decreases, ultimately vanishing when $\beta\alpha_- = 1$. Such a wave is called a **switch-off** slow compression wave since the transverse magnetic field H_y vanishes along the line $\beta\alpha_- = 1$, at which stage the compression is said to be **complete.** Since $\alpha_- < 1$, it then follows that the value of β attained at complete compression exceeds unity and so, from its definition, we see that in the complete compression state $a > b_x$. By taking the lower corresponding signs in equation (33.4), we also see that the velocity v_y increases across a slow compression wave. Recalling the definition of the magnetic pressure $p_m = \dfrac{\mu \boldsymbol{H}^2}{8\pi}$, we at once see that in a slow compression wave p_m decreases as the pressure p increases. When complete compression is attained, $p_m = 0$.

Alternatively, for a slow rarefaction wave starting from the point (α_0, β_0) in the (α, β)-plane β, and hence α_-, decrease. The integral curve through (α_0, β_0) then terminates on the line $\beta = 0$ representing the state of **complete rarefaction** in a slow wave. By using the definition of β, together with the polytropic gas law, we then see that the density ρ and the pressure p vanish on the line $\beta = 0$, corresponding to the occurrence of **cavitation.** Equation (31.7) shows that H_y increases across a slow rarefaction wave as β decreases, while equation (33.4) shows that the transverse velocity v_y also decreases. The magnetic pressure p_m increases across a slow rarefaction wave as the pressure p decreases; the complete rarefaction being determined by $p = 0$.

If the initial point (α_0, β_0) is taken on the line $\beta\alpha_- = 1$ then, by analogy with switch-off wave, the wave is called a **switch-on** slow rarefaction wave.

In a fast compression wave starting from similar constant physical conditions associated with a point (α_0, β_0) in the $(+)$ region of the (α, β)-plane and propagating in the positive x-direction β, and therefore α_+, increase and equation (31.7) shows that H_y increases. Consequently p and p_m increase, while from equation (33.4) we see that v_y decreases. Similar arguments show that in a fast rarefaction wave, as β decreases so also do H_y, p and p_m, while v_y increases. The magnetic field H_y and the magnetic pressure p_m vanish on the curve $\beta\alpha_+ = 1$ when the fast rarefaction is complete; however, the pressure p remains finite and so cavitation does not occur.

A simple modification of the above argument serves to indicate the variation of v_y with β when the fast and slow waves are propagating in the negative x-direction (see Example 6, § 37).

The behaviour of p, p_m and H_y are summarised in Fig. 17.

The arrows on the integral curves indicate the direction of change of β, while the quantities adjacent to these arrows show the variation of p, p_m and H_y with β. The symbol ↑ signifies an increase while the symbol ↓ signifies a decrease in the quantity to the left of the symbol.

The fact that the flow adjacent to a region of constant state is always a simple wave follows directly from their defining property that all the dependent variables are expressible in terms of one dependent variable, say ρ. This fact, when used with the property of characteristic curves that discontinuities in derivatives of dependent variables can take place across them, then shows that the straight

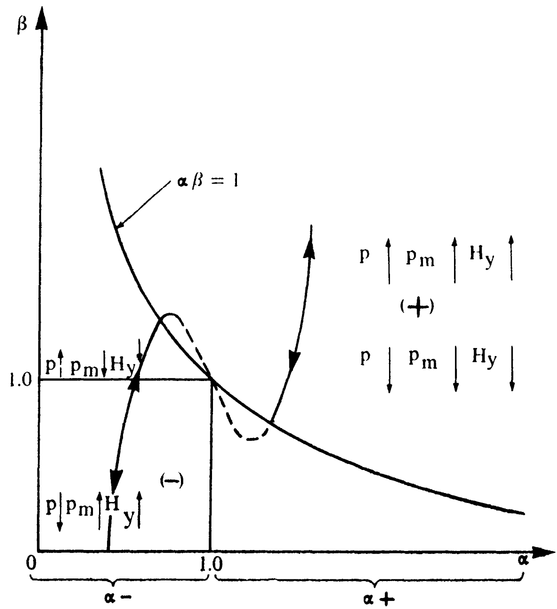

FIG. 17

characteristics bounding a constant state and a simple wave can be superimposed.

Having demonstrated that the physical variables in magnetoacoustic simple waves are all expressible in terms of the parameter β, we shall now use this to establish an important property of these simple waves. Since the development of a wave originating from a given initial state characterised by (α_0, β_0) is uniquely determined by the integral curve through the point (α_0, β_0), it follows that α and, consequently, c_n are functions of β (see Equation (31.5)). Thus the fast and slow families of characteristic curves $C^{(\pm f)}$ and $C^{(\pm s)}$, respectively, that are determined by equations (23.8), (25.32*a*) and (25.32*b*) can be written

$$\frac{dx}{dt} = v_x(\beta) \pm c_n(\beta), \tag{34.1}$$

where the $C^{(\pm f)}$ curves correspond to $c_n = c_f$ and the $C^{(\pm s)}$ curves correspond to $c_n = c_s$. The wave propagation speed dx/dt in magnetoacoustic simple waves is thus a function only of β.

Now a constant value $\beta = \beta^*$ specifies a constant state for the physical variables ρ, H_y, v_x and v_y and also causes the right-hand side of equation (34.1) to become constant. When c_n has been identified either with c_f or c_s and the sign has been chosen (the + sign signifying a wave advancing in the positive x-direction and the − sign a wave advancing in the negative x-direction), equation (34.1) can be integrated to give a family of straight lines. This family of straight line characteristics then describes fast or slow, forward or backward facing magnetoacoustic simple waves. Each of these characteristics has a different gradient, determined only by the value of $\beta = \beta^*$, and each represents a line of different constant physical state in the (x, t)-plane. The other three families of characteristics in a particular flow that are obtained by identifying c_n with the remaining three values of $\pm c_f$ and $\pm c_s$ will, in general, be curved. Hence our earlier assertion concerning the existence of a family of straight line characteristics, along each of which the physical state is constant, is proved.

When the right-hand side of equation (34.1) is such that characteristics corresponding to different values of β diverge with increasing time then the family of straight line characteristics in the (x, t)-plane appears as in Fig. 18(a).

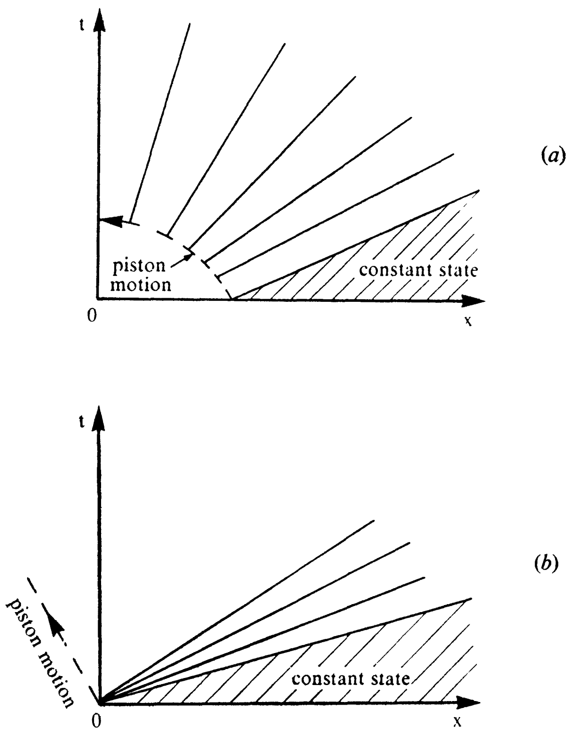

FIG. 18

Should the dependent variables in the magnetohydrodynamic equations be functions of the ratio $\xi = x/t$, instead of functions of x and t separately, then the family of straight line characteristics appears as in Fig. 18(*b*). These waves will be called **centred simple waves;** a name suggested by the behaviour of the characteristics at the origin.†

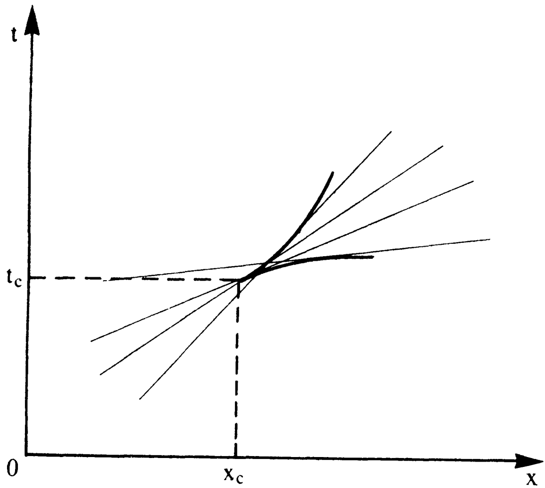

FIG. 19

Alternatively, when the right-hand side of equation (34.1) is such that characteristics corresponding to different values of β converge with increasing time, then the family of straight line characteristics appears as in Fig. 19. Since different constant physical states are associated with each different characteristic, the point of intersection of two characteristics must correspond to a point at which a simple

† Since solutions at different points in the (x, t)-plane are similar when x and t are increased in proportion, these flows are sometimes called **similarity flows.**

wave solution breaks down due to non-uniqueness. In the next chapter we shall see how this difficulty is resolved by introducing the notion of a shock wave across which the dependent variables experience finite jumps. In Fig. 19, this shock wave will form at the first point (x_c, t_c) at which the characteristics intersect.

§ 35. **The change of wave profile.** The general ideas of the final part of the previous section can be expressed more specifically in terms of compressive magnetoacoustic fast and slow simple waves advancing in the positive x-direction. In such compressive waves β, and hence α, increase. We have already seen that v_x increases with β, and so the wave propagation speed $\lambda = v_x+c$ also increases with β, because $c_n^2 = \alpha a^2$. To relate λ to a physical variable we now use the fact that $\beta = (\rho/\hat{\rho})^\gamma$, showing that λ is an increasing function of ρ. This is the situation illustrated in Fig. 19. The speed of propagation of different parts of fast and slow magnetoacoustic simple waves thus accelerate with increasing density causing the wave profiles to change their initial shapes and to steepen into fast and slow shocks. A similar argument applies to the formation of fast and slow shocks when the wave propagates in the negative x-direction.

When the waves are rarefaction waves, λ is a decreasing function of β or ρ, and the waves smooth out their initial profiles with increasing time. The change of profile of a typical physical variable is shown in Fig. 20 with the wave tending to a shock at $x = x_c$, $t = t_c$.

§ 36. **Elementary applications.** To illustrate the previous ideas we shall now consider some elementary applications.

(i) *Flow adjacent to a constant state* $(H_y = H_z = 0)$

Let us consider the flow adjacent to a constant state in which $H_x \neq 0$, $H_y = H_z = 0$, $\rho = \rho_0$, $v_x = v_{x0}$ and $v_y = v_{y0}$. In terms of the (α, β)-plane this state corresponds to a point either on the curve $\alpha\beta = 1$ or on the line $\alpha = 1$. (Why?) Consequently dv_y, as determined by equation (33.4), is infinite, as also is dH_y when determined by differentiation of equation (31.7) in conjunction with equation (31.14). This shows, as might be expected, that on these exceptional curves the method of solution in terms of α and β is not applicable.

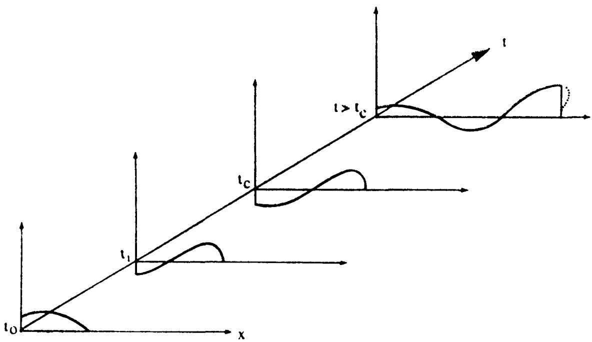

FIG. 20

To resolve this difficulty we need only to return to the original magnetoacoustic equations (29.1) to (29.3) and (29.5). These have the determinant (31.2) as their characteristic determinant and in the constant state in which $H_y = 0$ the first pair of equations becomes decoupled from the second pair of equations. The characteristic determinant has the characteristic roots $c_n = b_x$ and $c_n = a$. The root $c_n = b_x$ has already been examined in connection with transverse waves in § 30 (when, due to the rotation of the magnetic vector, we retained the component H_z) and so we shall consider only the root $c_n = a$. From equations (29.1) and (29.2) we obtain

$$dv_x = \pm \frac{\rho_0}{a_0} d\rho \tag{36.1}$$

while, since we shall assume that $b_x \neq a$, it follows directly from equations (29.3) and (29.5) that $dv_y = dH_y = 0$. The velocity component v_y thus retains its constant value v_{y0} across the flow region while the y-component of the magnetic field, H_y, remains zero. Hence we may omit the suffix zero from equation (36.1) which then describes the entire flow. There is thus no switch-on simple wave adjacent to a constant state region in which $H_y = H_z = 0$, and the integrated form of equation (36.1) leads directly to the ordinary gas dynamic Riemann invariants (see Example 6, § 28). Hence the flow adjacent to a constant state in which $H_y = H_z = 0$ is an ordinary gas simple wave.

(ii) *The simple piston problem*

A typical problem, in which gas is set in motion by mechanical means, is the gas flow induced by withdrawing towards the left a large plane piston from a long tube that is filled to the right with a perfectly electrically-conducting gas which is initially at rest. When there is no superimposed constant arbitrary magnetic field $\boldsymbol{H}_0$, the flow is an ordinary gas rarefaction wave. In such an ordinary gas flow the gas will expand and attempt to follow the piston, while the simple wave which advances into the gas at rest (the constant state region) will extend up to the piston face until cavitation occurs (see Example 7, § 37).

If the piston withdrawal velocity $V(t)$ is an increasing function of t, then, identifying the x-axis with the axis of the tube, the piston path in the (x, t)-plane is a curve similar to the dotted line in Fig. 18(a). Along each straight characteristic issuing from this curve the density and velocity are constant, their precise values being determined by the constant state conditions and by $V(t)$. Alternatively, if the piston is assumed to be withdrawn at a constant velocity V, then the expansion process is a centred simple wave as shown in Fig. 18(b) while the piston path in the (x, t)-plane is similar to the straight dotted line in that Figure.

Waves in which the fluid particles enter the simple wave region by crossing each straight characteristic from right to left are called **forward facing waves**. Thus the flows depicted in Figs. 18(*a*), (*b*), are forward facing rarefaction waves while the flow depicted in Fig. 19 is a forward facing compression wave. Conversely, if the fluid particles enter the simple wave region by crossing each straight characteristic from left to right, the wave is called a **backward facing wave.** Hence a backward facing rarefaction wave would result if gas filled the tube to the left of the piston, which was then withdrawn towards the right (how could a backward facing compression wave arise?).

However, when a general constant magnetic field is superimposed, the resulting flow becomes a magnetohydrodynamic flow involving magnetoacoustic rarefaction waves. Let us consider the specially simple problem that arises when the piston is withdrawn in the negative x-direction at a constant velocity V, while the magnetic field is transverse to the direction of piston motion in the gas at rest with density ρ_0. We take the y-axis in the direction of the magnetic field and the origin of the axes at the initial piston position.

The general families of $C^{(\pm)}$ characteristics are given by the equation

$$C^{(\pm)}: \quad \frac{dx}{dt} = v_x \pm c_n, \tag{36.2}$$

in which c_n is to be identified with one of the roots given in equations (25.31). As the magnetic field is purely transverse, $b_x = 0$, and it is immediately apparent from these equations that only the fast wave will propagate with

$$c_f = (a^2 + b_y^2)^{\frac{1}{2}}. \tag{36.3}$$

The wave, which will propagate into the gas filled region to the right, must be a forward facing rarefaction wave described by equation (36.2) in which we take the + sign and set $c_n = c_f = (a^2+b_y^2)^{\frac{1}{2}}$ to obtain

$$C^{(+)}: \quad \frac{dx}{dt} = v_x+(a^2+b_y^2)^{\frac{1}{2}}. \tag{36.4}$$

Now, setting $H_x = 0$ in equations (29.1) to (29.3) and (29.5) we see, by eliminating dv_x between equations (29.1) and (29.5), that

$$\frac{dH_y}{H_y} - \frac{d\rho}{\rho} = 0$$

or, in integral form,

$$H_y = k_1\rho, \tag{36.5}$$

where k_1 is a constant (cf., Example 12, § 28). Using the definition of b_y, and assuming the polytropic gas law (25.6), we can now rewrite equation (36.4) for the family of straight line $C^{(+)}$-characteristics in the form

$$\frac{dx}{dt} = v_x+\left(\gamma A\rho^{\gamma-1}+\frac{\mu k_1^2\rho}{4\pi}\right)^{\frac{1}{2}}. \tag{36.6}$$

The Riemann invariant relation across this $C^{(+)}$ family of characteristics is obtained by integrating equation (29.1), in which we have set $c_n = c_f$. The gas flow thus changes its properties along the $C^{(-)}$-characteristics and so, recalling from equations (25.32) that the + sign of $\lambda = v_x \pm c_n$ is to be associated with the − signs of the $\mp c_n$ terms in equations (29.1) to (29.3) and (29.5), we see that the Riemann invariant across the fast rarefaction wave is

$$r_- = v_x - \int_{\rho_0}^{\rho} \frac{c_f d\rho}{\rho}. \tag{36.7}$$

Since r_- is a constant, equal to its value r_{0-} in the constant state in which $v_{0x} = 0$, it follows that

$$v_x = \int_{\rho_0}^{\rho} \frac{c_f d\rho}{\rho}. \tag{36.8}$$

Now, the integrand can be written

$$\frac{c_f}{\rho} = K_1\rho^{-\frac{2}{3}}(1+K_2\rho^{\frac{1}{3}})^{\frac{1}{2}}, \tag{36.9}$$

where $K_1 = (\gamma A)^{\frac{1}{2}}$ and $K_2 = (\mu k_1^2/4\pi\gamma A)$ and, for $\gamma = 5/3$, can be integrated to give

$$v_x = 2\left(\frac{K_1}{K_2}\right)\{(1+K_2\rho^{\frac{1}{3}})^{\frac{3}{2}}-(1+K_2\rho_0^{\frac{1}{3}})^{\frac{3}{2}}\}. \tag{36.10}$$

This equation relates the x-component of velocity v_x to the density ρ and shows that cavitation will occur behind the piston ($\rho = 0$) should the piston withdrawal velocity V exceed the critical cavitation velocity † $(v_x)_{\text{cav}}$, given by

$$(v_x)_{\text{cav}} = 2\left(\frac{K_1}{K_2}\right)\{1-(1+K_2\rho_0^{\frac{1}{3}})^{\frac{3}{2}}\}. \tag{36.11}$$

When H_y/ρ is large, thus making K_2 large, $(v_x)_{\text{cav}}$ becomes

$$(v_x)_{\text{cav}} = -2K_1(K_2\rho_0)^{\frac{1}{2}}, \tag{36.12}$$

while when the magnetic field vanishes, corresponding to the ordinary gas dynamic case (see Example 7, § 37),

$$(v_x)_{\text{cav}} = -3K_1\rho_0^{\frac{1}{3}}$$

or,

$$(v_x)_{\text{cav}} = -3a_0. \tag{36.13}$$

If, as in § 34, the gradient dx/dt of a straight $C^{(+)}$ characteristic belonging to these fast centred rarefaction waves is denoted by ξ, then for $\gamma = 5/3$ equation (36.6) becomes

$$\xi = v_x+K_1\rho^{\frac{1}{3}}(1+K_2\rho^{\frac{1}{3}})^{\frac{1}{2}}. \tag{36.14}$$

† The velocity $(v_x)_{\text{cav}}$ at which cavitation occurs is often called the **escape speed.**

By using equation (36.10) in this result we can now obtain the relation between ρ and ξ:

$$\xi = 2\left(\frac{K_1}{K_2}\right)\{(1+K_2\rho^{\frac{1}{3}})^{\frac{3}{2}}-(1+K_2\rho_0^{\frac{1}{3}})^{\frac{3}{2}}\}+K_1\rho^{\frac{1}{3}}(1+K_2\rho^{\frac{1}{3}})^{\frac{1}{2}}. \tag{36.15}$$

Also, using equation (36.10) to eliminate ρ from equation (36.6), we find that

$$\xi = \frac{3v_x}{2} + \frac{K_1}{K_2}(1+K_2\rho_0^{\frac{1}{3}})^{\frac{3}{2}} - \frac{K_1}{K_2}\left\{\frac{v_x K_2}{2K_1} + (1+K_2\rho_0^{\frac{1}{3}})^{\frac{3}{2}}\right\}^{\frac{1}{3}}, \tag{36.16}$$

while the $C^{(+)}$-characteristics are determined by

$$\frac{dx}{dt} = \xi \text{ with } \xi = \frac{x}{t}. \tag{36.17}$$

It should be noticed that due to the assumption of infinite conductivity in the gas and to the configuration of the magnetic field it has been unnecessary to specify a magnetic boundary condition on the piston face. We mention here that in the general discussion of § 34 it was assumed that $H_x \neq 0$, from which then followed the conclusion that cavitation cannot occur in fast rarefaction waves. However, cavitation is possible in the case discussed in this Section because we have assumed that $H_x = 0$ and so the method of solution used in the discussion of § 34 breaks down.

(iii) *The generalised piston problem*

A generalisation of the previous problem is suggested by the special properties of magnetohydrodynamic contact discontinuities. In § 30 we saw that when $H_x \neq 0$ across a contact discontinuity, both the velocity and magnetic vectors must be continuous across the discontinuity. Consequently, if the fluid on one side of a plane contact discontinuity is replaced by a rigid perfectly conducting plane wall moving with velocity $\boldsymbol{u}$, the fluid and magnetic boundary conditions at the interface will be unchanged. Hence, across a perfectly conducting rigid wall bounding a perfectly conducting fluid the magnetic vector is continuous and the fluid in contact with the wall moves with the wall. Thus, unlike an ordinary hydrodynamic shear flow, the fluid sticks to the wall. We shall call this the generalised piston problem.

Take the x-axis normal to the wall and let the y and z-axes be fixed in the plane of the wall. Then, denoting the fluid velocity by $\boldsymbol{v}$, we have the boundary conditions

$$[\boldsymbol{B}] = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{v} = \boldsymbol{u},$$

or

$$B_{1x} = B_{2x}, \quad B_{1y} = B_{2y}, \quad B_{1z} = B_{2z} \tag{36.18a}$$

and

$$v_x = u_x, \quad v_y = u_y, \quad v_z = u_z, \tag{36.18b}$$

where the suffix 1 refers to a layer of material just inside the wall and the suffix 2 refers to a layer of fluid adjacent to the wall. Because of equations (36.18*a*) it is not necessary to distinguish components of $\boldsymbol{B}$ on either side of the interface.

This result is compatible with the electric boundary condition (15.2) when applied to two adjacent perfectly conducting media, the interface of which is penetrated by a continuous magnetic field vector $\boldsymbol{B}$. To see this it is only necessary to observe that the transverse electric field vanishes on either side of the perfectly conducting boundary across which $B_n \neq 0$ and on which $v_x = u_x$.

Indeed, the same boundary condition (15.2) provides the conditions to be satisfied across the interface between the fluid and a cavitation region that can occur when the perfectly conducting wall is withdrawn with an arbitrary velocity $\boldsymbol{u}$. First consider the situation in which cavitation just occurs at the wall. The cavitation condition is determined by $\rho = 0$ and, since both the fluid and the wall are perfect conductors, on which the transverse electric field vanishes, it follows from equation (15.2), the continuity of $\boldsymbol{B}$, and the fact that $\boldsymbol{t}$ is an arbitrary vector with $\boldsymbol{b} = \boldsymbol{n} \times \boldsymbol{t}$, that we must have

$$\boldsymbol{n} \times \{(\boldsymbol{u}-\boldsymbol{v}) \times \boldsymbol{B}\} = \boldsymbol{0}. \tag{36.19}$$

However, since the case when cavitation occurs immediately adjacent to the wall is only a limiting case of a finite cavitation zone, these conditions must also apply to a finite cavitation zone. The boundary conditions across the fluid-vacuum interface in the generalised piston problem are thus seen to be

$$\rho = 0, \tag{36.20a}$$

$$H_x(u_y-v_y)-H_y(u_x-v_x) = 0, \tag{36.20b}$$

$$H_x(u_z-v_z)-H_z(u_x-v_x) = 0. \tag{36.20c}$$

Motion of a rigid conducting wall with an arbitrary velocity $\boldsymbol{u}$ in an arbitrarily oriented unperturbed magnetic field $\boldsymbol{H}$ produces magnetohydrodynamic flows of considerable interest and complexity. The complexity arises from the fact that, unlike the simple piston problem, now slow waves and Alfvén waves can also propagate, and in general they will interact with each other. When, for example, the plane wall recedes from the gas in this more general flow, it is possible for a more general flow region to occur between the face of the wall and the simple wave region adjacent to the constant state, thereby complicating the solution.

Of this more complicated class of problems we shall only examine the qualitative features of one simple example. Consider the initial fluid motion that occurs when the wall is suddenly set in motion with velocity $u_y = -u_{y0}$, while the unperturbed magnetic field $\boldsymbol{H}$ is normal to the wall and is directed into the fluid which is initially at rest (i.e., $H_{x0} \neq 0$, $H_{y0} = H_{z0} = 0$). As the magnetic lines of force are frozen in the fluid, which because of boundary conditions (36.18) moves with the wall, it follows that a layer of fluid adjacent to the wall is displaced downwards, thereby producing a transverse magnetic field component H_y in the positive y-direction. As this component H_y is clearly a decreasing function of the distance x from the wall it produces a non-uniform transverse magnetic field which will, in turn, give rise to a current $\boldsymbol{j}$ in the fluid. If we denote the unit vectors in the x, y and z-directions by $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$, respectively, then from equation (2.9) we see that $\boldsymbol{j} = (c/4\pi)(\partial H_y/\partial x)\boldsymbol{e}_z$. However, H_y is a decreasing function of x and so the field H_y produces a current of strength $j_z = (c/4\pi)|\,\partial H_y/\partial x\,|$ in the negative z-direction. The effect on the fluid of the interaction of this current with H_y is then given by equation (6.1) which shows that a force of strength $(\mu/4\pi)H_y\,|\,\partial H_y/\partial x\,|$ acts in the positive x-direction. This force causes a wave to propagate out from the wall. The final state of this fluid motion is achieved when the wave has propagated to infinity, for then all the fluid is again at rest, but this time relative to the wall.

It can be shown that when the ratio of the sound speed to the Alfvén speed exceeds unity, or when u_{y0} is large, an ordinary gas shock wave (see Chapter 6) advances from the wall followed by a slow centred rarefaction wave. This combination of waves is such that the fluid changes from its initial undisturbed state to its final state of rest relative to the wall in crossing them. If the ratio of the sound speed to the Alfvén speed is less than unity then the flow conditions are resolved in a similar manner but with a magnetohydrodynamic switch-on shock wave taking the place of the ordinary gas shock wave. Large transverse magnetic fields will be produced by these mechanisms when u_{y0} is sufficiently large. The solution of this type of flow problem is sometimes called the **resolution of an initial magnetohydrodynamic shear flow discontinuity.**

(iv) *Generation and reflection of Alfvén waves*

It is not always necessary that magnetohydrodynamic wave motion should be initiated by mechanical means, as can be seen from the following example. Consider two horizontal parallel planes a distance d apart between which, in region 1, lies an incompressible perfectly conducting fluid of density ρ_1. Assume further that a constant magnetic field $\boldsymbol{H}_0$ is directed normal to these planes, and that the region 0 above them is a vacuum, while the region 2

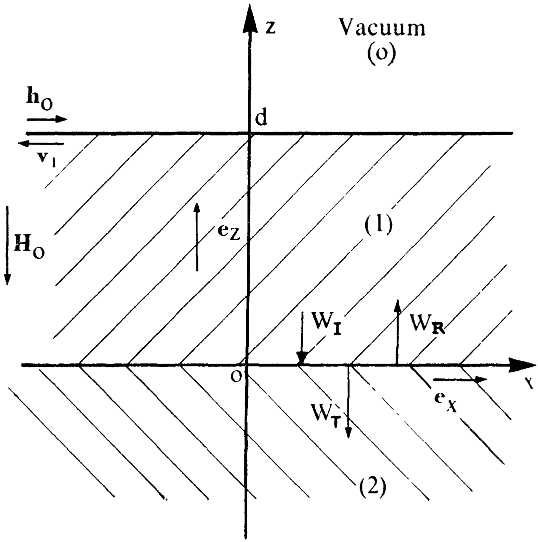

FIG. 21

below them is filled by another incompressible perfectly conducting fluid of density ρ_2. We shall neglect the effect of gravity and assume that the positive z-direction is parallel to $\boldsymbol{H}_0$, being directed upwards, with the origin at the interface between regions 1 and 2 (see Fig. 21). We take $\boldsymbol{H}_0$ in the opposite sense to z. Then, from the discussion of § 11, it is easy to see that if a small constant magnetic field $\boldsymbol{h}_0$ in the direction of the unit vector $\boldsymbol{e}_x$ parallel to the x-axis is suddenly created in the vacuum region, an Alfvén wave will propagate into region 1 with wavefront speed $b_1 = \sqrt{\dfrac{\mu H_0^2}{4\pi\rho_1}}$, thereby dispersing the initial surface current that was formed. The fluid particles in the Alfvén wave region will travel along the x-axis with speed

$$|v_{1x}| = h_0(\mu/4\pi\rho_1)^{\frac{1}{2}}$$

as shown by equation (11.7) (see Example 10, § 12). This Alfvén wave is an example of a magnetohydrodynamic wave disturbance that has been induced entirely magnetically without any external mechanical motion being imposed on the boundary.

It is interesting to examine how the Alfvén wave will be influenced when it encounters the interface $z = 0$ between the fluids of different density, since the situation there will be essentially similar to the special case studied in (iii) above. As in other branches of physics, in addition to the incident wave W_I, there will be a transmitted wave W_T entering region 2 and a reflected wave W_R returning through region 1. Since the boundary conditions (36.18) require the continuity of both the vectors $\boldsymbol{h}$ and $\boldsymbol{v}$ across the interface, the conditions to be satisfied at the interface must be

$$\boldsymbol{v}_T = \boldsymbol{v}_I + \boldsymbol{v}_R, \quad \boldsymbol{h}_\Gamma = \boldsymbol{h}_I + \boldsymbol{h}_R, \tag{36.21}$$

where the suffixes I, T, R denote the incident, transmitted and reflected waves, respectively. If the magnitudes of the vectors $\boldsymbol{h}$ and $\boldsymbol{v}$ are denoted by h and v, then for one-dimensional flow, equations (36.21) can be re-written in the scalar form

$$v_T = v_I + v_R, \quad h_T = h_I + h_R. \tag{36.21$'$}$$

Assuming that $\boldsymbol{h}_0$ is in the positive x-direction, the resulting current $\boldsymbol{j}$ in the fluid will flow in the negative y-direction (cf., the argument of (iii) above in connection with magnetohydrodynamic shear flow resolution). The force proportional to $\mu\boldsymbol{j} \times \boldsymbol{H}$ that acts to produce the fluid velocity $\boldsymbol{v}$ in the waves W_I and W_T will thus act in the negative x-direction causing the vectors $\boldsymbol{v}$ and $\boldsymbol{h}$ to be directed in opposite senses. For the reflected wave W_R propagating in the reverse direction the vector $\boldsymbol{v}$ reverses its direction and so $\boldsymbol{v}$ and $\boldsymbol{h}$ then have the same sense.

As the fluid velocity $\boldsymbol{v}$ is related to the magnetic vector $\boldsymbol{h}$ by the relation $\boldsymbol{h} = \pm \boldsymbol{v}(4\pi\rho/\mu)^{\frac{1}{2}}$, taking into account the sense of vectors $\boldsymbol{h}$ and $\boldsymbol{v}$, equations (36.21′) become

$$v_T = v_I + v_R,\quad v_T\rho_2^{\frac{1}{2}} = v_I\rho_1^{\frac{1}{2}} - v_R\rho_1^{\frac{1}{2}}.$$

These equations may be solved to give the following relationships between the amplitudes v_T and v_R of the transmitted and reflected waves in terms of the amplitude v_I of the incident wave,

$$\frac{v_R}{v_I} = \left(\frac{\rho_1^{\frac{1}{2}} - \rho_2^{\frac{1}{2}}}{\rho_1^{\frac{1}{2}} + \rho_2^{\frac{1}{2}}}\right),\quad \frac{v_T}{v_I} = \left(\frac{2\rho_1^{\frac{1}{2}}}{\rho_1^{\frac{1}{2}} + \rho_2^{\frac{1}{2}}}\right). \tag{36.22}$$

The corresponding relationships † between h_T, h_R and h_I follow directly from equations (36.22) and the relation $\boldsymbol{h} = \pm \boldsymbol{v}(4\pi\rho/\mu)^{\frac{1}{2}}$. From the relative senses of the vectors $\boldsymbol{v}$ and $\boldsymbol{h}$ in the waves W_I, W_T and W_R we have

$$\frac{h_R}{h_I} = -\frac{v_R}{v_I},\quad \frac{h_T}{h_I} = \frac{v_T}{v_I}\left(\frac{\rho_2}{\rho_1}\right)^{\frac{1}{2}}$$

giving

$$\frac{h_R}{h_I} = \left(\frac{\rho_2^{\frac{1}{2}} - \rho_1^{\frac{1}{2}}}{\rho_1^{\frac{1}{2}} + \rho_2^{\frac{1}{2}}}\right),\quad \frac{h_T}{h_I} = \frac{2\rho_2^{\frac{1}{2}}}{\rho_1^{\frac{1}{2}} + \rho_2^{\frac{1}{2}}}. \tag{36.23}$$

In deriving these results we have only considered the situation immediately after the incident wave reaches the interface. At a later time the reflected wave will itself be reflected back from the free surface located at the vacuum interface at $z = d$ to repeat the process. The

† Compare this situation with the reflection of waves at a density discontinuity on an elastic string. See, for example, Coulson, *Waves*, 1949, § 16.

general reflection boundary conditions at a free surface may be derived from these results by setting $\rho_2 = 0$ in equations (36.22) and (36.23). Similarly, the reflection boundary conditions at a rigid conducting wall may be derived by setting $\rho_2 = \infty$ in these equations.

These results have found application in the astrophysical problem of the dissipation of energy in the stratified atmosphere of the chromosphere. However, in this application the previous results must be slightly modified to include a gravitational potential and a fluid density that are dependent on the height z.

§ 37. Examples.

1. Consider the one-dimensional form of the magnetohydrodynamic characteristic equations. Show that when $H_x \neq 0$ across a contact surface the velocity and magnetic vectors are continuous across it, but that the density and entropy can experience finite jumps $d\rho$ and dS which are related by the expression

$$d\rho = \frac{-1}{a^2}\left(\frac{\partial p}{\partial S}\right) dS.$$

Show also that when $H_x = 0$, a shear flow can take place across the contact surface in which both the tangential velocity vector and the tangential magnetic vector can experience finite discontinuities in crossing the contact surface.

2. Assume the four one-dimensional characteristic equations determining magnetoacoustic simple waves and define α and β by the relations $\alpha = c_n^2/a^2$ and $\beta = a^2/b_x^2$. Show that

$$\frac{d\beta}{d\alpha} = \left(\frac{\gamma}{2-\gamma}\right)\frac{(\alpha^2\beta-1)}{\alpha^2(\alpha-1)}.$$

Determine the fast and slow wave regions in the (α, β)-plane and deduce the curve on which the maxima and minima of β lie.

3. Derive the expression for the generalised magneto-acoustic Riemann invariants:

$$K_{\pm} = \beta \mid \alpha_{\pm} - 1 \mid^{-\gamma^*} \pm \gamma^* \int_{\alpha_0}^{\alpha} \alpha_{\mp}^{-2} \mid \alpha_{\pm} - 1 \mid^{-(1+\gamma^*)} d\alpha_{\pm},$$

and show also that

$$dv_x = \varepsilon\gamma^{-1}\hat{a}\alpha_{\pm}^{\frac{1}{2}}\beta^{-\frac{1}{2}(1+(1/\gamma))}d\beta.$$

By using the equation

$$H_y^2 = (\alpha - 1)(\beta - \alpha^{-1})H_x^2,$$

show that we may write H_y/H_x in the form

$$\frac{H_y}{H_x} = \operatorname{sgn}\left(\frac{H_{y0}}{H_x}\right)\sqrt{\frac{(\alpha_{\pm} - 1)(\alpha_{\pm}\beta - 1)}{\alpha_{\pm}}},$$

where H_{y0} is the value of H_y in the constant state adjacent to the simple wave region (H_x is constant and we assume $H_{y0} \neq 0$).

Use the fact that H_y only vanishes on the line $\alpha\beta = 1$ in order to show that H_y does not change its sign across slow and fast waves, and hence that

$$\operatorname{sgn}\left(\frac{H_y}{H_x}\right) = \operatorname{sgn}\left(\frac{H_{y0}}{H_x}\right) = \operatorname{sgn}(H_{y0}H_x).$$

Then, using the expression for dv_x, prove that

$$dv_y = \mp\varepsilon\gamma^{-1}\hat{a}\beta^{-\frac{1}{2}(1+(1/\gamma))}\sqrt{\frac{(\alpha_{\pm} - 1)}{(\alpha_{\pm}\beta - 1)}}\operatorname{sgn}(H_{y0}H_x)d\beta.$$

4. By setting $p = a^2/c_n^2$ and $r = b_x^2/c_n^2$, and by using the equation

$$H_y^2 = (\alpha - 1)(\beta - \alpha^{-1})H_x^2.$$

show that

$$\frac{dr}{dp} = \left(\frac{1}{2-\gamma}\right)\left[\frac{2r}{p} + \frac{\gamma r(r-1)}{(p-1)}\right].$$

Sketch the integral curves of this equation and show that the points (0, 0) and (1, 1) in the (p, r)-plane are nodes of the equation. Deduce the fast and slow wave regions in the (p, r)-plane. The point (1, 0) is also a singularity of the equation and is called a **saddle point.** Show that in the neighbourhood of this saddle point the integral curves behave locally like rectangular hyperbolae. Notice that at nodes, the exponents λ_i of the localised solution have identical signs, whereas at the saddle point the signs differ.

5. Prove that the wave investigated in Example 12 of § 28 must be a fast wave. Show, by considering the definition of the slow wave speed c_s in terms of a, b and b_x, that the slow wave has zero velocity. By combining the characteristic equation corresponding to the final equation of Example 12 of § 28 with the characteristic equation

$$\mp c_n d\rho + \rho dv_x = 0,$$

which is derived from the continuity equation, show that

$$dv_x = \pm \rho^{\frac{1}{2}(\gamma-3)}\left(\gamma A + \frac{\mu k_1^2}{4\pi}\,\rho^{(2-\gamma)}\right)^{\frac{1}{2}} d\rho,$$

where the square of the sound speed $a^2 = \gamma A \rho^{\gamma-1}$. Hence prove that the generalised x-Riemann invariants $r_\pm$ for the velocity v_x are

$$r_\pm = v_x \pm \int_0^\rho t^{\frac{1}{2}(\gamma-3)}\left(\gamma A + \frac{\mu k_1^2}{4\pi}\,t^{(2-\gamma)}\right)^{\frac{1}{2}} dt.$$

Notice that the factor k_1 provides a measure of the influence of the magnetic effects on the flow and that when $k_1 = 0$ the generalised Riemann invariants reduce to the ordinary gas dynamic invariants (cf. Example 6, § 28).

6. Prove that when $\varepsilon = 1$, v_x varies in the same sense as p, ρ and β, while when $\varepsilon = -1$, v_x varies in the opposite sense to p, ρ and β. Prove also that, when $\varepsilon \operatorname{sgn}(H_x H_{y0}) > 0$, v_y and p_m vary in the opposite sense, while when $\varepsilon \operatorname{sgn}(H_x H_{y0}) < 0$, v_y and p_m vary in the same sense.

7. Prove that in an ordinary gas rarefaction wave produced by suddenly withdrawing a plane piston from a tube filled with gas at rest, the Riemann invariant across the forward facing centred rarefaction wave that results is

$$v_x + \frac{2(a_0 - a)}{(\gamma - 1)} = 0,$$

where a_0 is the velocity of sound in the constant state. By setting $dx/dt = \xi (= x/t)$, prove that

$$v_x = \left(\frac{2}{\gamma+1}\right)(\xi - a_0)$$

and that the cavitation velocity is

$$(v_x)_{\text{cav}} = \frac{-2a_0}{\gamma - 1}.$$

Deduce the regions in the (x, t)-plane that are occupied by the constant state region, the simple wave region and the cavitation zone (if the piston withdrawal velocity is sufficiently fast) and sketch the behaviour of the $C^{(-)}$ characteristics.

8. Consider a fast centred rarefaction wave in which the magnetic field $\boldsymbol{H}$ is very strong and is purely transverse to the direction of flow. Using the fact that $H_y = k_1\rho$ and assuming the gas pressure law $p = A\rho^{5/3}$ prove that the $C^{(+)}$ family of characteristics is determined by the equation

$$\frac{dx}{dt} = v_x + K_1\sqrt{K_2\rho},$$

where $K_1 = (5A/3)^{1/2}$ and $K_2 = (3\mu k_1^2/20\pi A)$. Setting $dx/dt = \xi$ show that

$$\rho = \frac{1}{9}\left(\frac{\xi}{K_1 K_2^{1/2}} + 2\rho_0^{1/2}\right)^2$$

and that

$$v_x = \tfrac{2}{3}(\xi - K_1\sqrt{K_2\rho_0}).$$

Deduce the escape velocity of the gas.

CHAPTER VI

MAGNETOHYDRODYNAMIC SHOCK WAVES

§ 38. **General considerations.** In our discussion of simple waves in a perfect gas we have already seen that under certain conditions the wave profiles of the dependent variables (density, pressure, etc.) can steepen, until at a certain time they develop an infinite gradient. In Fig. 20 this process is suggested by the waveform at time $t = t_c$. The change in the nature of the solution beyond this time has already been indicated in principle by the arguments used in connection with Fig. 19. These demonstrated that the solution was not unique at time t_c, and we show later that it is sufficient that this non-uniqueness should take the form of an ordinary jump discontinuity in the dependent variables when crossing the wavefront. A wave of this type, in which the dependent variables experience finite jumps across the wavefront, is called a **shock wave.** From definition (13.1) it can be seen that shock waves are **strong discontinuities,** whereas the wavefronts of Chapters IV and V that are described by characteristics are **weak discontinuities.**

This situation is obviously a mathematical idealisation of a physical process involving real gases with dissipative effects, in which large changes in the physical variables occur within a very thin region of the flow. Experimental results show that the thickness of a shock wave in a real gas is of the order of a few mean free paths, so a mathematical idealisation of a shock wave in which the dependent variables experience a finite jump across a geometrical surface represents a good approximation to reality. We shall neglect the dissipative effects of viscosity and electrical resistivity that must be included if the actual shock structure is to be studied and again use the approximation provided by the Lundquist equations.

In order to determine the relationship between the dependent variables on adjacent sides of a magnetohydrodynamic shock we shall first use the fact that the Lundquist equations can be decomposed into a simple system of equations, each having the form of a conservation law. Indeed, equation (6.5) already has this form since it expresses the conservation of mass. Then, by integrating these equations over a volume moving with the shock and using a theorem which we shall now prove, the Gauss divergence theorem can be applied to the result to determine the jump conditions across the shock.

However, before establishing an important rate of change theorem for a volume integral, in which the bounding surface is moving, let us first recall that the general differential form of a conservation law is

$$\frac{\partial U}{\partial t} + \operatorname{div} \boldsymbol{F} = G. \tag{38.1}$$

By integrating this equation over a volume element dV and using the definition of the divergence operator † it is easy to interpret this equation physically. We find that the sum of the rate of change of the amount of a scalar U contained in a volume element dV and the flux of the vector $\boldsymbol{F}$ into dV, together equal the contribution from the source distribution G throughout dV in a unit time. As we have already remarked, we shall have occasion to identify the Cartesian components of the Lundquist equations with laws of this form.

Let us now prove a general theorem that is vital to the

† See Rutherford, *Vector Methods*, 1954, § 51.

study of shock waves. We have seen that in a shock wave we must consider the possibility of discontinuities across a moving surface, which we now call the **shock front.** So, to do this, we start by considering an arbitrary surface $S(t)$ moving with velocity $\boldsymbol{q}$, that bounds a volume $V(t)$ in which a differentiable scalar function U is defined.

Write

$$I = \int_V U dV, \tag{38.2}$$

and notice that in time increment δt, the integrand of I becomes, to the first order,

$$U + \left(\frac{\partial U}{\partial t}\right)\delta t.$$

However, during the time increment δt the volume bounded by $S(t)$ changes as $V(t)$ moves. To calculate the effect of this change we use the fact that the vector surface element $d\boldsymbol{S}$ of $S(t)$ moves a distance $\boldsymbol{q}\delta t$ in the time increment δt, and so the corresponding element of volume change is $\boldsymbol{q}\,.\,d\boldsymbol{S}\delta t$. The corresponding increment in the integrand of I due to this is thus $U\boldsymbol{q}\,.\,d\boldsymbol{S}\delta t$. So, combining these results, we find that

$$I + \delta I = \int_{V(t)} \left\{U + \left(\frac{\partial U}{\partial t}\right)\delta t\right\} dV + \int_{S(t)} U\boldsymbol{q}\,.\,d\boldsymbol{S}\delta t. \tag{38.3}$$

Subtracting equation (38.2) from this result, dividing by δt and taking the limit as $\delta t \to 0$ we have finally proved the following volume rate of change theorem,

$$\frac{D}{Dt}\int_{V(t)} U dV = \int_{V(t)} \frac{\partial U}{\partial t} dV + \int_{S(t)} U\boldsymbol{q}\,.\,d\boldsymbol{S}. \tag{38.4}$$

This important theorem is just the three-dimensional statement of a familiar theorem concerning differentiation under the integral sign † (see Example 1, § 44). By applying

† See Gillespie, *Integration*, 1947, § 41.

the Gauss divergence theorem to equation (38.4) we can express it in the alternative form

$$\frac{D}{Dt}\int_{V(t)} U dV = \int_{V(t)} \left(\frac{\partial U}{\partial t} + \operatorname{div}(U\boldsymbol{q})\right) dV. \quad (38.5)$$

Let us now see how this theorem may be used to derive the jump conditions that are permitted by a conservation law of the form (38.1). We assume that a discontinuity surface exists and that an arbitrary part of it, $S^*(t)$, divides the volume $V(t)$ into volumes $V_0(t)$ and $V_1(t)$, and the surface $S(t)$ into surfaces $S_0(t)$ and $S_1(t)$, respectively. The value of functions on adjacent sides of, and arbitrarily close to, $S^*(t)$ will be denoted by the suffixes 0, 1 according as $S^*(t)$ is approached from $V_0(t)$ or $V_1(t)$, respectively.

Using the expression for $\partial U/\partial t$ given by equation (38.1) in theorem (38.5), applying the Gauss divergence theorem, and assuming first that neither U nor G have any singularities we find that

$$\frac{D}{Dt}\int_{V(t)} U dV = \int_{S(t)} (U\boldsymbol{q} - \boldsymbol{F}).d\boldsymbol{S} + \int_{V(t)} G dV. \quad (38.6)$$

So, subtracting from this equation the corresponding equations in which $V(t)$ is identified, respectively, with $V_0(t)$ and $V_1(t)$, we find that

$$\int_{S^*(t)} (U\boldsymbol{q} - \boldsymbol{F})_0 . d\boldsymbol{S}_0^* + \int_{S^*(t)} (U\boldsymbol{q} - \boldsymbol{F})_1 . d\boldsymbol{S}_1^* = 0, \quad (38.7)$$

where $d\boldsymbol{S}_i^*$ is the outward directed vector surface element of $S^*(t)$ with respect to $V_i(t)$. However, since

$$d\boldsymbol{S}_0^* = -d\boldsymbol{S}_1^* = \boldsymbol{n} dS^*,$$

say, where $\boldsymbol{n}$ is the outward drawn normal to $S^*(t)$ with respect to $V_0(t)$ and, furthermore, $S^*(t)$ is an arbitrary part of a discontinuity surface, it follows at once from equation

(38.7) that

$$(U\boldsymbol{q}-\boldsymbol{F})_0 . \boldsymbol{n}-(U\boldsymbol{q}-\boldsymbol{F})_1 . \boldsymbol{n} = 0. \qquad (38.8)$$

If $[X]$ denotes the jump X_1-X_0 in the quantity X across $S^*(t)$, then equation (38.8) may be written in the alternative form

$$[\tilde{\lambda}U-\boldsymbol{F}.\boldsymbol{n}] = 0, \qquad (38.9)$$

where

$$\tilde{\lambda} = \boldsymbol{q}.\boldsymbol{n} \qquad (38.10)$$

is the normal speed of propagation of the discontinuity surface.

A special case of interest to us occurs when the source term G of equation (38.1) is a divergence div $\boldsymbol{g}$ of some vector $\boldsymbol{g}$, which is also discontinuous across $S^*(t)$. The volume integral in the last term of equation (38.6) can then also be transformed by the Gauss divergence theorem into a surface integral when the same argument as before then yields the jump equation

$$[\tilde{\lambda}U-\boldsymbol{F}.\boldsymbol{n}]+[\boldsymbol{g}.\boldsymbol{n}] = 0. \qquad (38.11)$$

Equation (38.11) is the compatibility condition to be satisfied by jumps in the terms U, $\boldsymbol{F}$ and G of the conservation law (38.1) across each element of area of a general curved discontinuity surface moving with local normal velocity $\tilde{\lambda} = \boldsymbol{q}.\boldsymbol{n}$. When, as we shall do later, we consider plane discontinuity surfaces (plane shocks), a single jump condition of the form (38.11) will be uniformly valid over the entire discontinuity surface for each conservation equation involved.

§ **39. Magnetohydrodynamic shocks.** In order that we may utilise the results of the previous section to determine the jump relations that are permitted in magnetohydrodynamic shocks it is necessary that the Lundquist equations be displayed in conservation form. Equation (6.5) is already in conservation form

$$\frac{\partial \rho}{\partial t} + \text{div}\,(\rho \boldsymbol{v}) = 0. \tag{6.5}$$

To reduce the momentum equation (7.5′) to conservation form we first transform it to its equivalent form (10.18), and then use the solenoidal property of $\boldsymbol{H}$ expressed in equation (2.3), together with the mass conservation law (6.5) above, to write it in the component form

$$\frac{\partial(\rho v_x)}{\partial t} + \text{div}\,(\rho v_x \boldsymbol{v}) = -\frac{\partial}{\partial x}\left(p + \frac{\mu \boldsymbol{H}^2}{8\pi}\right) + \frac{\mu}{4\pi}\,\text{div}\,(H_x \boldsymbol{H}), \tag{39.1a}$$

$$\frac{\partial(\rho v_y)}{\partial t} + \text{div}\,(\rho v_y \boldsymbol{v}) = -\frac{\partial}{\partial y}\left(p + \frac{\mu \boldsymbol{H}^2}{8\pi}\right) + \frac{\mu}{4\pi}\,\text{div}\,(H_y \boldsymbol{H}), \tag{39.1b}$$

$$\frac{\partial(\rho v_z)}{\partial t} + \text{div}\,(\rho v_z \boldsymbol{v}) = -\frac{\partial}{\partial z}\left(p + \frac{\mu \boldsymbol{H}^2}{8\pi}\right) + \frac{\mu}{4\pi}\,\text{div}\,(H_z \boldsymbol{H}). \tag{39.1c}$$

Using the solenoidal condition (2.3) and expanding the right-hand side of the equation (5.1′) for the magnetic field enables it to be written in the component form

$$\frac{\partial H_x}{\partial t} + \text{div}\,(H_x \boldsymbol{v}) = \text{div}\,(v_x \boldsymbol{H}), \tag{39.2a}$$

$$\frac{\partial H_y}{\partial t} + \text{div}\,(H_y \boldsymbol{v}) = \text{div}\,(v_y \boldsymbol{H}), \tag{39.2b}$$

$$\frac{\partial H_z}{\partial t} + \text{div}\,(H_z \boldsymbol{v}) = \text{div}\,(v_z \boldsymbol{H}). \tag{39.2c}$$

The energy conservation equation (8.6), from which the entropy equation (8.2′) was derived, is also in the required form and for a fluid with no dissipative effects reduces to

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho v^2+\rho e+\frac{\mu H^2}{8\pi}\right)+\operatorname{div}\left\{v\left(\tfrac{1}{2}\rho v^2+\rho e+\frac{\mu H^2}{8\pi}\right)\right\}$$

$$= -\operatorname{div}\left(pv+\frac{\mu H^2}{8\pi}v-\frac{\mu(v\,.\,H)}{4\pi}H\right). \qquad (39.3)$$

If, now, we apply result (38.11) to these equations we find the following jump conditions:

mass conservation

$$[\tilde{\lambda}\rho-\rho v\,.\,n] = 0, \qquad (39.4)$$

momentum conservation

$$[\tilde{\lambda}\rho v-\rho v(v\,.\,n)] = \left[p+\frac{\mu H^2}{8\pi}\right]n-\frac{\mu}{4\pi}[(H\,.\,n)H], \qquad (39.5)$$

magnetic field

$$[\tilde{\lambda}H-(v\,.\,n)H] = -[(H\,.\,n)v], \qquad (39.6)$$

energy conservation

$$\left[\tilde{\lambda}\left(\tfrac{1}{2}\rho v^2+\rho e+\frac{\mu H^2}{8\pi}\right)-\left(\tfrac{1}{2}\rho v^2+\rho e+\frac{\mu H^2}{8\pi}\right)(v\,.\,n)\right]$$

$$=\left[p(v\,.\,n)+\frac{\mu H^2}{8\pi}(v\,.\,n)-\frac{\mu(v\,.\,H)}{4\pi}(H\,.\,n)\right], \qquad (39.7)$$

solenoidal jump condition

$$[H\,.\,n] = 0. \qquad (39.8)$$

It is useful to display these results in a different form by expressing them in terms of

$$\tilde{v}_n = v\,.\,n-\tilde{\lambda}, \qquad (39.9)$$

the normal fluid velocity component relative to the normal velocity $\tilde{\lambda}$ of the discontinuity surface. Alternatively expressed, the jump conditions become:

mass conservation

$$[\rho\tilde{v}_n] = 0, \tag{39.4'}$$

momentum conservation

$$[\rho\tilde{v}_n\boldsymbol{v}+p^*\boldsymbol{n}] = \frac{\mu}{4\pi}[H_n\boldsymbol{H}], \tag{39.5'}$$

magnetic field

$$[\tilde{v}_n\boldsymbol{H}-H_n\boldsymbol{v}] = \boldsymbol{0}, \tag{39.6'}$$

energy conservation

$$\left[\tilde{v}_n\left(\tfrac{1}{2}\rho v^2+\rho e+\frac{\mu H^2}{8\pi}\right)+v_n p^*-\frac{\mu(\boldsymbol{v}\,.\,\boldsymbol{H})}{4\pi}H_n\right] = 0 \tag{39.7'}$$

solenoidal jump condition

$$[H_n] = 0, \tag{39.8'}$$

where $p^* = p+\dfrac{\mu H^2}{8\pi}$ is the total pressure and the suffix n again denotes the normal component.

Equation (39.4′) expresses the simple fact that the mass flow through the discontinuity surface is constant. We shall denote the mass flow through the surface by m, and thus

$$m = \rho_0\tilde{v}_{n0} = \rho_1\tilde{v}_{n1}, \tag{39.10}$$

where the suffixes 0 and 1 denote opposite sides of the discontinuity surface. A discontinuity surface will only be called a *shock* when the mass flow m through the surface is non-zero. Hence fluid particles must cross a shock front.

As may be expected from the study of boundary conditions in Chapter II, the solenoidal jump condition (39.8′) shows that H_n is continuous across a shock front. We shall later use this fact, together with (39.10), to remove H_n and m from within the square brackets of the jump conditions.

The jump conditions (39.4′) to (39.8′) have been derived quite generally for an element of an arbitrary shock front moving with local normal velocity $\tilde{\lambda} = \boldsymbol{q}.\boldsymbol{n}$. Henceforth we shall assume that the shock propagation is steady and that the shock front (discontinuity surface) is plane, so that these jump conditions become uniformly true across the entire shock front for all time. The constant value that we attribute to $\tilde{\lambda}$ will determine how the shock moves relative to our reference frame. If, for example, we set $\tilde{\lambda} = 0$ the shock will be stationary, whereas if we set $\tilde{\lambda} = -\tilde{v}_{n0}$, then $v_{n0} = 0$ and the shock will propagate with speed $\tilde{v}_{n0}$ into the gas of region 0 which is at rest.

§ **40. The generalised Hugoniot condition.** Let us display the energy equation (39.7′) in a form which has special significance in ordinary fluid dynamics. To do this we first write it in the form

$$\tfrac{1}{2}m[\boldsymbol{v}^2]+m[e]+\frac{m\mu}{8\pi}[\tau \boldsymbol{H}^2]+[v_n p]+\frac{\mu}{8\pi}[v_n \boldsymbol{H}^2] - \frac{\mu H_n}{4\pi}[\boldsymbol{v}.\boldsymbol{H}] = 0, \quad (40.1)$$

where $\tau = 1/\rho$ is the specific volume of the fluid. However, from the thermodynamical considerations of § 8, we have already seen that $R = c_p - c_v$ and the adiabatic exponent $\gamma = c_p/c_v$, while for a polytropic gas $e = c_v T$, and so equation (8.8) becomes

$$e = \frac{p\tau}{\gamma - 1}. \quad (40.2)$$

The energy equation (40.1) thus becomes

$$\tfrac{1}{2}m[\boldsymbol{v}^2]+\frac{m}{\gamma-1}[p\tau]+\frac{m\mu}{8\pi}[\tau \boldsymbol{H}^2]+[v_n p]+\frac{\mu}{8\pi}[v_n \boldsymbol{H}^2] - \frac{\mu H_n}{4\pi}[\boldsymbol{v}.\boldsymbol{H}] = 0. \quad (40.3)$$

The first term of this equation can now be replaced by the expression that is obtained by forming the scalar product of equation (39.5′) with $\langle \boldsymbol{v} \rangle$, where $\langle Q \rangle = \frac{1}{2}(Q_0 + Q_1)$ denotes the average value of Q. The transformed equation obtained from (40.3) is

$$\frac{m}{\gamma-1}[p\tau]+[v_n p]-[p]\langle v_n\rangle+\frac{m\mu}{8\pi}[\tau H^2]+\frac{\mu}{8\pi}[v_n H^2]$$
$$-\frac{\mu H_n}{4\pi}[\boldsymbol{v}\,.\,\boldsymbol{H}]-\frac{\mu}{8\pi}[H^2]\langle v_n\rangle+\frac{\mu H_n}{4\pi}[\boldsymbol{H}]\,.\,\langle \boldsymbol{v}\rangle = 0. \quad (40.4)$$

Now, since the speed $\tilde{\lambda}$ of the discontinuity surface must obviously be continuous across the discontinuity surface, it follows at once from equation (39.9) that

$$[\tilde{v}_n] = [v_n] \quad (40.5)$$

or, alternatively, that

$$m[\tau] = [v_n]. \quad (40.6)$$

So, applying the identity $[PQ] = \langle P\rangle[Q]+\langle Q\rangle[P]$ to the second and third terms of equation (40.4), using equation (40.6) and re-writing the remaining terms containing the magnetic field vector we find that

$$\frac{m}{\gamma-1}[p\tau]+m\langle p\rangle[\tau]+\frac{m\mu}{8\pi}[\tau H^2]+\frac{\mu}{8\pi}[v_n H^2]$$
$$-\frac{\mu H_n}{4\pi}[\boldsymbol{v}\,.\,\boldsymbol{H}]-\frac{\mu}{8\pi}[H^2]\langle v_n\rangle+\frac{\mu H_n}{4\pi}[\boldsymbol{H}]\,.\,\langle \boldsymbol{v}\rangle = 0. \quad (40.7)$$

Expanding terms of the form $[PQ]$ in equation (40.7) and using the identity $\langle P^2\rangle - \langle P\rangle^2 \equiv \frac{1}{4}[P]^2$ together with the result

$$m\langle\tau\rangle[\boldsymbol{H}]\,.\,\langle \boldsymbol{H}\rangle + m[\tau]\langle \boldsymbol{H}\rangle^2 = H_n[\boldsymbol{v}]\,.\,\langle \boldsymbol{H}\rangle, \quad (40.8)$$

obtained by forming the scalar product of equation (39.6′) with $\langle \boldsymbol{H}\rangle$, we find that, provided $m \neq 0$, equation (40.7) simplifies to

$$\frac{1}{\gamma-1}[p\tau]+\langle p\rangle[\tau]+\frac{\mu}{16\pi}[\tau][\boldsymbol{H}]^2=0. \qquad (40.9)$$

When this expression is re-written in the form

$$[e+\langle p\rangle\tau]=-\frac{\mu}{16\pi}[\tau][\boldsymbol{H}]^2, \qquad (40.10)$$

the left-hand side of the equation is seen to be the **Hugoniot relation** of ordinary gas dynamics.† For this reason equation (14.10) will be called the **generalised Hugoniot relation.** When $\boldsymbol{H} \equiv \boldsymbol{0}$ this equation only involves thermodynamical quantities and reduces to the results of ordinary gas dynamics.

However, when $\boldsymbol{H} \neq \boldsymbol{0}$ but is normal to the plane of the shock the conducting fluid behaves as an ordinary fluid for then, as $\boldsymbol{H} = H_n\boldsymbol{n}$ and we have seen that H_n is continuous across the shock front, we must have $[\boldsymbol{H}] = \boldsymbol{0}$.

By introducing the ratio $r = \tau_0/\tau_1 (= \rho_1/\rho_0)$ equation (40.9) can be used to express the pressure ratio p_1/p_0 across the shock front in the form

$$\frac{p_1}{p_0}=\frac{(\gamma+1)r-(\gamma-1)}{(\gamma+1)-(\gamma-1)r}+\frac{\mu[\boldsymbol{H}]^2}{8\pi p_0}\frac{(\gamma-1)(r-1)}{(\gamma+1)-(\gamma-1)r}. \qquad (40.11)$$

This important equation illustrates an ambiguity in the results obtained so far; namely, that the pressure ratio p_1/p_0 depends on the density ratio r across the shock, which so far may be either less than or greater than unity. Obviously in any physical situation the direction of the pressure and density jump across the shock front is uniquely determined, and so we must next examine how the physically permissible range of r may be determined.

That the mathematical solution obtained from the Lundquist equations has this non-uniqueness should be of

† See Rutherford, *Fluid Dynamics*, 1949, § 46.

no surprise to us since we already encountered a similar non-uniqueness in § 34 in connection with Fig. 19. In the next section we shall show how this non-uniqueness may be resolved by using thermodynamical considerations.

§ 41. The compressive nature of magnetohydrodynamic shocks. We have just seen that the jump conditions (39.4′) to (39.8′) which relate values on adjacent sides of a discontinuity surface do not determine the senses of the jumps involved (i.e., the increase or decrease). Since in a physical situation a solution must be unique, it is clear that some extra condition must be imposed on the jump conditions so that only physically real jump conditions are allowed. To achieve this it will be necessary to supplement the jump conditions by the thermodynamical requirement that the *entropy cannot decrease across a shock front.* It should be clearly understood that this supplementary condition is imposed from outside the framework of magnetohydrodynamics, and that it is in fact implied by the second law of thermodynamics.

First, let us choose the direction of the normal $\boldsymbol{n}$ to the shock front so that

$$\tilde{v}_n = v_n - \tilde{\lambda} > 0, \tag{41.1}$$

and denote quantities on the side of the shock front into which $\boldsymbol{n}$ is directed by the suffix 1 and quantities on the other side by the suffix 0. Fluid particles will thus leave region 0 and cross the shock front to enter into region 1. It is conventional to refer to the side of the shock front through which the fluid enters as the **front** of the shock or the side **ahead** of the shock. The other side is called the **back** of the shock or the side **behind** the shock.

We shall now prove that when used with the jump conditions, the second law of thermodynamics, which requires the entropy not to decrease on crossing the shock, so that

$$S_1 \geqq S_0, \tag{41.2}$$

also implies that $\rho_1 > \rho_0$ and $p_1 > p_0$. That is the second law of thermodynamics imposes the requirement that only compressive shocks are allowed.

Let us start by noticing that equations (8.11) and (8.12) imply that,

$$S_1 - S_0 = c_v \log\left(\frac{p_1 \tau_1^\gamma}{p_0 \tau_0^\gamma}\right) \tag{41.3}$$

or, alternatively, that

$$S_1 - S_0 = c_v \log\left(\frac{p_1}{p_0}\right) - \gamma c_v \log r, \tag{41.4}$$

where, again $r = \tau_0/\tau_1$. Now by setting $k^2 = \mu[\boldsymbol{H}]^2/8\pi p_0$, which is always a non-negative number, we may express equation (40.11) in the form

$$\frac{p_1}{p_0} = \frac{(1+k^2)\{(\gamma+1)r-(\gamma-1)\}-2k^2 r}{(\gamma+1)-(\gamma-1)r}, \tag{41.5}$$

showing that

$$\frac{p_1}{p_0} < (1+k^2)\left\{\frac{(\gamma+1)r-(\gamma-1)}{(\gamma+1)-(\gamma-1)r}\right\}. \tag{41.6}$$

Because the pressure ratio is inherently positive, the numerator and denominator of inequality (41.6) must both be of the same sign, while the fact that $\gamma > 1$ shows that

$$\frac{\gamma-1}{\gamma+1} < r < \frac{\gamma+1}{\gamma-1}. \tag{41.7}$$

We must now show that the entropy condition (41.2) disallows density ratios r less than unity.

Since $k^2 \geqq 0$, we shall prove our proposition if we succeed in showing that when p_1/p_0 in equation (41.4) is replaced by $\{(\gamma+1)r-(\gamma-1)\}/\{(\gamma+1)-(\gamma-1)r\}$, the entropy condition implies that $r > 1$.

So we shall now consider the expression

$$S_1 - S_0 = c_v \log\left\{\frac{(\gamma+1)r-(\gamma-1)}{(\gamma+1)-(\gamma-1)r}\right\} - \gamma c_v \log r. \quad (41.8)$$

Assuming the state 0 ahead of the shock to be a fixed state we can write $r = r(S_1)$ and it is easily established that

$$\frac{dS_1}{dr} = \frac{c_v\gamma(\gamma^2-1)(r-1)^2}{r\{(\gamma+1)r-(\gamma-1)\}\{(\gamma+1)-(\gamma-1)r\}}. \quad (41.9)$$

Now the numerator is positive, and we have already seen from the arguments leading to the inequalities (41.7) that both factors in brackets in the denominator have the same sign, and so

$$\frac{dS_1}{dr} > 0. \quad (41.10)$$

Consequently, since from condition (41.2) the entropy cannot decrease across the shock, while condition (41.10) shows that S_1 and r change in the same sense, we have proved that $r(S_1)$ increases across a magnetohydrodynamic shock. As $r(S_0) = 1$ we have thus also proved that $r > 1$ and hence that *magnetohydrodynamic shocks are compressive.* Equation (40.11) then shows that $p_1/p_0 > 1$ across the shock front as was asserted at the start. Inequality (41.7) must now be modified to

$$1 < r < \frac{\gamma+1}{\gamma-1}. \quad (41.11)$$

The mass conservation jump condition (39.4′) shows that

$$\tilde{v}_0 = r\tilde{v}_1, \quad (41.12)$$

and so, relative to the shock front, the gas ahead of the shock moves faster than the gas behind the shock. Let us now introduce the notions of Mach number and subsonic and supersonic flow. The local **Mach number** $M = v/a$ of an ordinary gas flow is defined to be the ratio of the local gas speed v and the local speed of sound a. In a general flow the Mach number will obviously be a function of position, while in the steady flow across a shock such as we are considering here it will have different constant values on opposite sides of the shock front. A flow will be said to be **subsonic** when its Mach number $M<1$ and it will be said to be **supersonic** when its Mach number $M>1$.

When the shock is stationary ($\tilde{\lambda} = 0$) and the magnetic field acts normal to the shock front, result (41.12) can be shown to be equivalent to the statement that the flow ahead of an ordinary stationary gas dynamic shock is supersonic and the flow behind the shock is subsonic † (see Example 5, § 44).

We can prove a analogous result for a stationary perpendicular magnetohydrodynamic shock in which the magnetic field acts normal to the direction of flow. The result will be derived by a modification of the argument that was used to derive the Hugoniot relation. To simplify our notation let us identify the normal $\boldsymbol{n}$ with the direction of the x-axis so that $H_n = 0$ and $v_n = v_x = |\boldsymbol{v}|$. We shall denote the transverse magnetic field strength by H_t. Equations (39.4′) and (39.7′) then become

$$\frac{\rho_1}{\rho_0} = \frac{v_{x0}}{v_{x1}}, \tag{41.13}$$

$$\rho_0 v_{x0}^2 + p_0 + \frac{\mu H_{t0}^2}{8\pi} = \rho_1 v_{x1}^2 + p_1 + \frac{\mu H_{t1}^2}{8\pi}, \tag{41.14}$$

$$\frac{v_{x0}}{v_{x1}} = \frac{H_{t1}}{H_{t0}}, \tag{41.15}$$

$$\tfrac{1}{2}v_{x0}^2 + \frac{\gamma p_0 \tau_0}{\gamma - 1} + \frac{\mu H_{t0}^2 \tau_0}{4\pi} = \tfrac{1}{2}v_{x1}^2 + \frac{\gamma p_1 \tau_1}{\gamma - 1} + \frac{\mu H_{t1}^2 \tau_1}{4\pi}. \tag{41.16}$$

† See Rutherford, *Fluid Dynamics*, 1959, § 46.

Equations (41.13) and (41.15) immediately show that

$$r = \frac{\rho_1}{\rho_0} = \frac{\tau_0}{\tau_1} = \frac{H_{t1}}{H_{t0}} = \frac{v_{x0}}{v_{x1}}, \tag{41.17}$$

thereby enabling equations (41.14) and (41.16) to be written in the form

$$M_0^2\left(1-\frac{1}{r}\right) = \frac{1}{\gamma}\left\{\left(\frac{p_1}{p_0}-1\right) + \frac{\mu H_{t0}^2}{8\pi p_0}(r^2-1)\right\} \tag{41.18}$$

and

$$M_0^2\left(1-\frac{1}{r^2}\right) = \frac{\mu H_{t0}^2(r-1)}{2\pi\gamma p_0} + \frac{2}{\gamma-1}\left\{\frac{1}{r}\left(\frac{p_1}{p_0}\right)-1\right\}, \tag{41.19}$$

when the relation $a_0^2 = \gamma p_0 \tau_0$ is used. Since the shocks must be compressive, and so $r > 1$, the elimination of the ratio p_1/p_0 between these equations † leads to the quadratic equation

$$(2-\gamma)\frac{\mu H_{t0}^2}{8\pi p_0}r^2 + \gamma\left\{\frac{\mu H_{t0}^2}{8\pi p_0} + \tfrac{1}{2}(\gamma-1)M_0^2 + 1\right\}r$$
$$-\tfrac{1}{2}\gamma(\gamma+1)M_0^2 = 0. \tag{41.20}$$

Consequently, since $r > 1$, equation (41.20) yields the inequality

$$\tfrac{1}{2}\gamma(\gamma+1)M_0^2 > (2-\gamma)\frac{\mu H_{t0}^2}{8\pi p_0} + \gamma\left\{\frac{\mu H_{t0}^2}{8\pi p_0} + \tfrac{1}{2}(\gamma-1)M_0^2 + 1\right\},$$

provided that $\gamma < 2$ ($\gamma = 5/3$ for a plasma). Or, more simply, the inequality

$$v_x^2 > b_{t0}^2 + a_0^2, \tag{41.21}$$

where b_{t0}, the Alfvén speed ahead of the shock, is given by

$$b_{t0}^2 = \frac{\mu H_{t0}^2}{4\pi \rho_0}. \tag{41.22}$$

† The elimination of M_0^2 would lead directly to the alternative form of the generalised Hugoniot relation expressed in equation (40.11).

Since the Alfvén speed is a more proper speed by which to classify magnetohydrodynamic disturbances, we define the **Alfvén number** A, by analogy with the Mach number, to be the ratio of the fluid speed v to the Alfvén speed b giving

$$A = \frac{v}{b}. \tag{41.23}$$

Inequality (41.21) then shows that

$$A_0^2 > 1 + \left(\frac{a_0}{b_{t0}}\right)^2, \tag{41.24}$$

and so the flow ahead of a perpendicular magnetohydrodynamic shock is **super-Alfvénic** relative to the conditions ahead of the shock front. A similar argument establishes that the flow behind the shock front is **sub-Alfvénic** relative to the conditions behind the shock front. The ratio $v_x/(b_t^2 + a^2)^{\frac{1}{2}}$ is often called the **magnetic Mach number.**

The situation appropriate to a moving plane shock wave may be easily deduced from the jump conditions (39.4′) to (39.8′) by setting $\tilde{\lambda} = U$, where U is the constant shock velocity. Alternatively the result may be deduced directly from the above equations by super-imposing a constant velocity $-U$ on the system, with U directed along the x-axis.

§ 42. Magnetohydrodynamic shock wave classification. We have already seen that under certain conditions a continuous wave motion can tend to a shock wave. This fact, coupled with the identification of the different types of continuous wave motions which we have called fast and slow waves, transverse waves and entropy waves, suggests that an analogous classification for shock waves might exist. This is indeed the case, and to establish such a classification for magnetohydrodynamic shock waves we first use equations (39.8′) and (39.10) to write the jump relation (39.5′) in the form

$$m[\boldsymbol{v}]+\frac{[p]}{[\tau]}[\tau]\boldsymbol{n}+\frac{\mu}{4\pi}\langle\boldsymbol{H}\rangle.[\boldsymbol{H}]\boldsymbol{n}-\frac{\mu H_n}{4\pi}[\boldsymbol{H}]=\boldsymbol{0}. \quad (42.1)$$

Then, using the scalar equation

$$m[\tau]-[v_n] = 0, \quad (40.6)$$

that was derived from the continuity of the shock front velocity $\tilde{\lambda}$, the jump relation (39.6′) becomes

$$H_n[\boldsymbol{v}]-\langle\boldsymbol{H}\rangle[v_n]-m\langle\tau\rangle[\boldsymbol{H}] = \boldsymbol{0}. \quad (42.2)$$

The two vector equations (42.1,2) and the scalar equation (40.6) then represent a set of seven homogeneous scalar equations for seven scalar jump quantities; namely, the six scalar components of $[\boldsymbol{v}]$ and $[\boldsymbol{H}]$ and the scalar quantity $[\tau]$.

For these equations to be consistent, and for a non-trivial solution to exist, the determinant of the coefficients of these jump quantities must vanish. It is a straightforward matter (see Example 7, § 44) to show that the vanishing of this determinant yields the equation

$$\langle\tau\rangle^2 m\left\{\langle\tau\rangle m^2-\frac{\mu H_n^2}{4\pi}\right\}\left\{\langle\tau\rangle m^4+(\langle\tau\rangle[\tau]^{-1}[p]-\mu\langle\boldsymbol{H}\rangle^2)m^2 -[\tau]^{-1}[p]\frac{\mu H_n^2}{4\pi}\right\}=0. \quad (42.3)$$

This may be regarded either as an equation for the mass flux m through the shock front, or as an equation for the shock velocity. Indeed, by writing equation (41.1) in the form $m\tau = v_n-\tilde{\lambda}$, and averaging across the shock front we find that

$$\tilde{\lambda} = \langle v_n\rangle-m\langle\tau\rangle. \quad (42.4)$$

The vanishing of the different factors of equation (42.3) thus corresponds to different modes of magnetohydrodynamic shock wave propagation.

(a) *Fast and slow shocks*

Let us consider the vanishing of the last factor of equation (42.3). The factor

$$\langle\tau\rangle m^4+(\langle\tau\rangle[\tau]^{-1}[p]-\mu\langle \boldsymbol{H}\rangle^2)m^2-[\tau]^{-1}[p]\,\frac{\mu H_n^2}{4\pi}=0$$

can be written as

$$(m^2+[\tau]^{-1}[p])\left(m^2-\langle\tau\rangle^{-1}\,\frac{\mu H_n^2}{4\pi}\right)$$

$$=\frac{m^2\mu\langle\tau\rangle^{-1}}{4\pi}\,(\langle \boldsymbol{H}\rangle^2-H_n^2). \qquad (42.5)$$

This is a quadratic equation in m^2 and will thus have two roots, the smaller of which we shall denote by m_s^2 and the larger by m_f^2. As the right-hand side of the equation is positive it follows directly that each factor on the left-hand side must be of the same sign, giving

$$m_s^2 \leqq -[\tau]^{-1}[p] \leqq m_f^2, \qquad (42.6)$$

and

$$m_s^2 \leqq \mu H_n^2/(4\pi\langle\tau\rangle) \leqq m_f^2. \qquad (42.7)$$

We notice that the thermodynamical considerations of § 41 which proved that $[p]>0$ across a shock wave also proved that $[\tau]<0$, and so the middle term in inequality (42.6) is positive, as is the middle term of inequality (42.7). The roots m_f and m_s of equation (42.5) describe the mass flow (or the shock velocity) of **fast** and **slow** magnetohydrodynamic shock waves. The inequalities (42.6,7) provide the justification for these names.

Using equation (42.5) and the jump relations (39.4′) to (39.8′), the permissible jumps of quantities across magnetohydrodynamic fast and slow shocks are seen to be

$$[\boldsymbol{H}]=\varepsilon_{f,s}m^2(\langle \boldsymbol{H}\rangle-H_n\boldsymbol{n}), \qquad (42.8)$$

$$[\boldsymbol{v}] = \varepsilon_{f,\,s} m\left(\frac{\mu H_n}{4\pi}\langle \boldsymbol{H}\rangle - \langle \tau\rangle m^2 \boldsymbol{n}\right), \tag{42.9}$$

$$[\tau] = -\varepsilon_{f,\,s}\left(\langle \tau\rangle m^2 - \frac{\mu H_n^2}{4\pi}\right), \tag{42.10}$$

where $\varepsilon_{f,\,s}$ is a parameter characterising the strength of the jump across a fast (f) or a slow (s) shock.

Now, as $[\boldsymbol{H}^2] = 2\langle \boldsymbol{H}\rangle . [\boldsymbol{H}]$, it follows from (42.8) that

$$[\boldsymbol{H}]^2 = 2\varepsilon_{f,\,s} m^2 (\ \boldsymbol{H}\rangle^2 - H_n^2), \tag{42.11}$$

and so, taking the scalar product of equation (39.5′) with $\boldsymbol{n}$ and using this result yields,

$$[p] = \varepsilon_{f,\,s} m^2\left(\langle \tau\rangle m^2 - \frac{\mu\langle \boldsymbol{H}\rangle^2}{4\pi}\right). \tag{42.12}$$

Finally, eliminating $\varepsilon_{f,\,s}$ between equations (42.10) and (42.11), we find that

$$[\boldsymbol{H}^2] = -2m^2[\tau]\{\langle \boldsymbol{H}\rangle^2 - H_n^2\} \Big/ \left\{\langle \tau\rangle m^2 - \frac{\mu H_n^2}{4\pi}\right\}. \tag{42.13}$$

Since $[\tau] < 0$ across a shock wave, this equation and inequality (42.7) together imply that the magnetic field strength *increases* across a fast shock and *decreases* across a slow shock. As would have been expected from jump condition (39.8′), equation (42.8) shows that the magnetic field experiences a tangential jump discontinuity on crossing the shock front.

Expanding equation (42.8) shows that

$$\boldsymbol{H}_{t1} = \left(\frac{1 + \frac{1}{2}\varepsilon_{f,\,s} m^2}{1 - \frac{1}{2}\varepsilon_{f,\,s} m^2}\right)\boldsymbol{H}_{t0},$$

where the suffix t denotes the transverse component. Consequently

$$[\boldsymbol{H}] = [\boldsymbol{H}_t] = \left(\frac{\varepsilon_{f,\,s} m^2}{1 - \frac{1}{2}\varepsilon_{f,\,s} m^2}\right)\boldsymbol{H}_{t0},$$

showing that the jump $[\boldsymbol{H}]$ is parallel to $\boldsymbol{H}_{t0}$, while equation (42.8) itself shows that the sense of the jump is the same as that of the tangential component of $\langle \boldsymbol{H} \rangle$. These results, together with the fact that it can be shown from shock stability considerations † that the magnetic field cannot reverse its direction, thus allow us to make the following statement. The tangential component of the magnetic field retains its direction across fast and slow shocks, increasing its magnitude across a fast shock and decreasing its magnitude across a slow shock.

Analogous to switch-on and switch-off simple waves we can also have **switch-on** and **switch-off shocks**. The passage of a switch-on shock results in the appearance of a magnetic field when one was previously absent ahead of the shock, and conversely for a switch-off shock. It is apparent from our discussion of the behaviour of $[\boldsymbol{H}]$ that a switch-on shock must be a fast shock and a switch-off shock must be a slow shock.

As in the case of ordinary gas dynamics, a fast or slow magnetohydrodynamic shock is uniquely determined by prescribing all the quantities ahead of it and the normal relative gas velocity $\tilde{v}_{n1} = v_{n1} - \tilde{\lambda}$ or the pressure $p_1(>p_0)$ behind it.

(b) *Transverse shocks*

The vanishing of the second factor of equation (42.3) gives the result

$$m = \pm \left(\frac{\mu H_n^2}{4\pi \langle \tau \rangle} \right)^{\frac{1}{2}}. \tag{42.14}$$

Disturbances of this type are called **transverse shock waves** and it follows directly from the jump equations (39.4′) to

† Although it is mathematically possible for the magnetic field to *reverse* its direction across a slow shock, it can be shown that such a slow shock wave would be unstable to perturbations of the initial conditions, and so it represents a non-physical shock.

(39.8′) and equation (42.14) that in transverse shock waves

$$[\boldsymbol{H}] = \varepsilon m\langle \boldsymbol{H}\rangle \times \boldsymbol{n}, \tag{42.15}$$

$$[\boldsymbol{v}] = \varepsilon \frac{\mu H_n}{4\pi} \langle \boldsymbol{H}\rangle \times \boldsymbol{n}, \tag{42.16}$$

$$[\tau] = 0, \tag{42.17}$$

$$[p] = 0. \tag{42.18}$$

Hence the jumps $[\boldsymbol{H}]$ and $[\boldsymbol{v}]$ are parallel to one another and lie in the plane of the transverse shock, while the pressure and density remain constant across it. Combining equations (42.15) and (42.16) and using the fact that the density remains constant yields the relation

$$[\boldsymbol{v}] = \pm [\boldsymbol{H}] \sqrt{\frac{\mu}{4\pi\rho}}, \tag{42.19}$$

which exhibits the same relationship between $\boldsymbol{v}$ and $\boldsymbol{H}$ as does equation (11.7) which was obtained in our preliminary discussion of Alfvén waves.

Equation (42.15) implies that

$$[\boldsymbol{H}^2] = 2\langle \boldsymbol{H}\rangle . [\boldsymbol{H}] = 0, \tag{42.20}$$

and so the strength of the magnetic field is unchanged across a transverse shock; the magnetic field simply rotates on crossing the plane of the shock.

Equations (42.17) and (42.18) together with equation (41.3) show that

$$[S] = 0. \tag{42.21}$$

There is no classical hydrodynamical discontinuity that corresponds to this type of wave since, when $m \neq 0$ in ordinary hydrodynamical flows, it follows from the continuity of the pressure that every other quantity must be continuous.

(c) *Contact discontinuities*

The vanishing of the only remaining factor in equation (42.3), namely

$$m = 0, \tag{42.22}$$

gives rise to a **contact discontinuity** which, by its definition in terms of equation (42.22), has no flow across the discontinuity surface.

It follows directly from the jump relations (39.4′) to (39.8′) that when $H_n \neq 0$,

$$[\boldsymbol{H}] = \mathbf{0}, \tag{42.23}$$

$$[\boldsymbol{v}] = \mathbf{0}, \tag{42.24}$$

$$[p] = 0, \tag{42.25}$$

while $[\tau]$ may be arbitrary. Since no fluid crosses a contact discontinuity this last result does not violate the thermodynamical requirements that must be satisfied by shocks and it is, in fact, an ordinary hydrodynamical type of contact discontinuity.

However, when $H_n = 0$, the same reasoning then shows that

$$[p^*] = \left[p + \frac{\mu H^2}{8\pi}\right] = 0, \tag{42.26}$$

while the tangential components of the jumps $[\boldsymbol{H}]$ and $[\boldsymbol{v}]$ may be arbitrary.

(d) *Weak shocks*

It is easy to show that as the strong discontinuities in $\boldsymbol{H}$, $\boldsymbol{v}$, τ and p tend to weak discontinuities, and so $[\boldsymbol{H}]$, $[\boldsymbol{v}]$, $[\tau]$ and $[p]$ tend to zero, so equation (42.3) for strong discontinuities tends, apart from a factor, to the characteristic equation (25.30) for weak discontinuities. To achieve this result we use the fact that although $\boldsymbol{H}$, $\boldsymbol{v}$, $\tau = 1/\rho$ and p are continuous across the discontinuity surface, their first derivatives are discontinuous. Consequently we must make the following changes of notation to obtain our result:

$$[\boldsymbol{v}]\to\delta\boldsymbol{v},\quad [\tau]\to-\frac{1}{\rho^2}\,\delta\rho,\quad [\boldsymbol{H}]\to\delta\boldsymbol{H}$$

and

$$m\to\mp\rho c_n,\quad \langle\tau\rangle\to\frac{1}{\rho},\quad -\frac{[p]}{[\tau]}\to\rho^2a^2,\quad \langle\boldsymbol{H}\rangle\to\boldsymbol{H},$$

where δ denotes the jump in the normal derivative of the quantity associated with it on crossing the wavefront. This result also implies the important result that shock waves and weak discontinuities propagate at different speeds.

Since for a weak shock the density ratio can be written

$$r = 1+\varepsilon,$$

where ε is small and positive, it follows directly from equation (41.9) that

$$\frac{dS_1}{d\varepsilon}\simeq\tfrac{1}{4}c_v\gamma(\gamma^2-1)\varepsilon^2.$$

Consequently, integrating, we find that

$$S_1-S_0 = \tfrac{1}{12}c_v\gamma(\gamma^2-1)\varepsilon^3,$$

showing that the entropy change S_1-S_0 across a weak shock is of third order with respect to the change in r. As equation (41.8) was itself chosen to provide an upper bound to the entropy increase across a shock, irrespective of the magnetic field, it also follows that the entropy increase across a shock is not greater than third order with respect to the change in $\boldsymbol{H}$.

§ 43. Magnetohydrodynamic shock stability. It is important to understand that so far no consideration has been given to the stability of the magnetohydrodynamic shocks that have been discussed. By this we mean that although these shocks are mathematically possible, it still needs to be shown that they are stable with respect to perturbations of the initial conditions, and so are also physically realisable.

This stability criterion is suggested by the work of Hadamard who, at the turn of the century in connection with studies of wave motion, suggested that in the physical world all real solutions depend continuously and boundedly on initial data. Consequently, if a mathematically possible solution is found to be unstable with respect to perturbations of the initial data, then it cannot represent a physical solution. Indeed, if such a wave could be initiated, the first irregularity of flow velocity or density that the wave encountered would cause it to break up into a combination of stable waves.

Although we shall not pursue these ideas further, it is nevertheless possible to apply them to magnetohydrodynamic shocks in order to determine under exactly what conditions our shock solutions represent physically realisable shocks. Arguments of this type show, for example, that the tangential component of the magnetic field cannot reverse its direction when crossing a slow shock. Also, that unlike classical hydrodynamics, a magnetohydrodynamic shear flow discontinuity is stable provided the magnetic fields in the fluids adjacent to the discontinuity have no normal component at the interface. In this latter case the magnetic field actually exerts a stabilising effect on the flow. A physical reason for this is that when $H_n = 0$, perturbations of the interface distort the fluid and stretch the magnetic lines of force which, as we saw in § 11, behave like elastic strings and so help to reduce the perturbation. This effect is absent from a classical hydrodynamic shear flow which is consequently unstable and rapidly degenerates into turbulent flow.

Further discussion of this topic will not be possible here but perhaps something of the importance of such an analysis can be appreciated from the brief comments that have been offered.

§ 44. Examples

1. Show that the one-dimensional form of the vector rate of change theorem

$$\frac{D}{Dt}\int_{V(t)} U dV = \int_{V(t)} \frac{\partial U}{\partial t} dV + \int_{S(t)} U\boldsymbol{q}\,.\,d\boldsymbol{S}$$

for a scalar function $U = U(x, t)$, where $\boldsymbol{q}$ is the velocity of the surface $S(t)$ bounding volume $V(t)$, is

$$\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} U(x, t)dx = \int_{\alpha(t)}^{\beta(t)} \frac{\partial U}{\partial t} dx + U(\beta, t)\frac{d\beta}{dt} - U(\alpha, t)\frac{d\alpha}{dt},$$

where α and β are functions of the single real variable t. Now assume that a plane shock wave in an ordinary gas occurs at a point $x = \xi(t)$ in a tube of gas of arbitrary length whose ends at any time t coincide with the planes $x = \alpha(t)$ and $x = \beta(t)$. Consider a gas with density ρ, velocity u, pressure p and specific internal energy e and denote quantities ahead of the shock by the suffix 0 and quantities behind the shock by the suffix 1. Let $v_i = u_i - U$ denote the gas speed relative to the shock which moves with speed $U = \dfrac{d\xi}{dt}$ and let $m = \rho_0 v_0$ be the mass flow through side 0 of the shock. Apply the above theorem to the ordinary gas dynamic conservation equations of mass, momentum and energy that describe the flow in the two columns of length $\alpha(t) \leqq x \leqq \xi(t) - \varepsilon$ and $\xi(t) + \varepsilon \leqq x \leqq \beta(t)$, where ε is arbitrarily small and positive, to show that as $\alpha(t) \to \xi(t) - \varepsilon$ and $\beta(t) \to \xi(t) + \varepsilon$ and $\varepsilon \to 0$, so the conservation equations reduce to the ordinary gas shock conditions,

mass conservation:

$$\rho_0 v_0 = \rho_1 v_1,$$

momentum conservation:

$$mu_0 + p_0 = mu_1 + p_1,$$

or

$$\rho_0 v_0^2 + p_0 = \rho_1 v_1^2 + p_1,$$

energy conservation:

$$m(\tfrac{1}{2}u_0^2 + e_0) + u_0 p_0 = m(\tfrac{1}{2}u_1^2 + e_1) + u_1 p_1.$$

2. Express the Lundquist equations in the form of general conservation laws of the type

$$\frac{\partial U}{\partial t} + \operatorname{div} \boldsymbol{F} = \operatorname{div} \boldsymbol{g},$$

where U is a scalar and $\boldsymbol{F}$ and $\boldsymbol{g}$ are vectors.

3. Use the results of the previous example and the jump equation

$$[\tilde{\lambda} U - \boldsymbol{F}.\boldsymbol{n}] + [\boldsymbol{g}.\boldsymbol{n}] = 0,$$

to derive the magnetohydrodynamic jump relations, where $\tilde{\lambda} = \boldsymbol{q}.\boldsymbol{n}$ is the normal speed of the discontinuity surface assumed to be moving with the arbitrary velocity $\boldsymbol{q}$.

4. Prove the generalised Hugoniot relation

$$\left(\frac{1}{\gamma - 1}\right)[p\tau] + \langle p \rangle [\tau] + \frac{\mu}{16\pi}[\tau][\boldsymbol{H}]^2 = 0.$$

Use the equation of state $p\tau = RT$, and the fact that magnetohydrodynamic shocks are compressive, to prove that the temperature behind a shock is greater than the temperature ahead of a shock.

5. By assuming that the magnetic field is perpendicular to a stationary shock front, show that in an ordinary gas shock the flow is subsonic relative to conditions behind the shock front and supersonic relative to conditions ahead of the shock front.

6. Prove that the flow behind a perpendicular stationary magnetohydrodynamic shock is sub-Alfvénic relative to conditions behind the shock front.

7. By choosing the x-axis parallel to the normal $\boldsymbol{n}$ to a plane magnetohydrodynamic shock front show that the characteristic determinant associated with the jump relations across the shock front reduces to

$$\langle\tau\rangle^2 m\left\{\langle\tau\rangle m^2-\frac{\mu H_n^2}{4\pi}\right\}\left\{\langle\tau\rangle m^4+(\langle\tau\rangle[\tau]^{-1}[p]\right.$$
$$\left.-\mu\langle\boldsymbol{H}\rangle^2)m^2-[\tau]^{-1}[p]\frac{\mu H_n^2}{4\pi}\right\}=0.$$

8. Derive the jump conditions across fast and slow magnetohydrodynamic shocks, transverse shocks and contact discontinuities.

9. By aligning the x-axis with the normal $\boldsymbol{n}$ to a plane magnetohydrodynamic shock and using the correspondences

$$[\boldsymbol{v}]\to\delta\boldsymbol{v},\quad [\tau]\to-\frac{1}{\rho^2}\delta\rho,\quad [\boldsymbol{H}]\to\delta\boldsymbol{H}$$

and

$$m\to\mp\rho c_n,\quad \tau\to\frac{1}{\rho},\quad -\frac{[p]}{[\tau]}\to\rho^2a^2,\quad \langle\boldsymbol{H}\rangle\to\boldsymbol{H},$$

prove that the shock jump relations transform to the one-dimensional characteristic equations for weak discontinuities.

CHAPTER VII

STEADY MAGNETOHYDRODYNAMIC FLOW

§ 45. Ordinary gas dynamic characteristics in steady flow. The notion of characteristic curves can easily be extended to include steady flow configurations. Before examining steady isentropic magnetohydrodynamic flow, let us briefly consider the steady isentropic flow of an ordinary gas. The appropriate equations can be derived from the first two Lundquist equations by setting $\boldsymbol{H} \equiv \mathbf{0}$ and ignoring the terms involving $\partial/\partial t$, when we find that for a polytropic gas

$$\operatorname{div}(\rho \boldsymbol{v}) = 0, \tag{45.1}$$

$$(\boldsymbol{v}\,.\,\operatorname{grad})\boldsymbol{v} + \frac{a^2}{\rho}\operatorname{grad}\rho = 0. \tag{45.2}$$

Let us consider two-dimensional flow in the (x, y)-plane and assume that spatial discontinuities (characteristics), if they exist, are represented by the equation

$$\phi(x, y) = \text{constant}, \tag{45.3}$$

where $\phi = 0$ represents the particular discontinuity which we wish to examine (see Fig. 22). Then, following the reasoning of § 23, let us transform equations (45.1, 2) from the (x, y)-plane to the (ϕ, y')-plane in which

$$y' = y. \tag{45.4}$$

The solution in the (ϕ, y')-plane will then be continuous across $\phi = 0$ with respect to y' and its derivatives, but discontinuous with respect to derivatives involving ϕ. The transformation can then be accomplished by using the identities

$$\frac{\partial}{\partial x} \equiv \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}, \quad \frac{\partial}{\partial y} \equiv \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} + \frac{\partial}{\partial y'}. \tag{45.5}$$

Equation (45.1) becomes

$$\rho\left(\frac{\partial \phi}{\partial x}\frac{\partial v_x}{\partial \phi} + \frac{\partial \phi}{\partial y}\frac{\partial v_y}{\partial \phi}\right) + \left(v_x \frac{\partial \phi}{\partial x} + v_y \frac{\partial \phi}{\partial y}\right)\frac{\partial \rho}{\partial \phi} + \rho\frac{\partial v_y}{\partial y'} + v_y \frac{\partial \rho}{\partial y'} = 0. \tag{45.6}$$

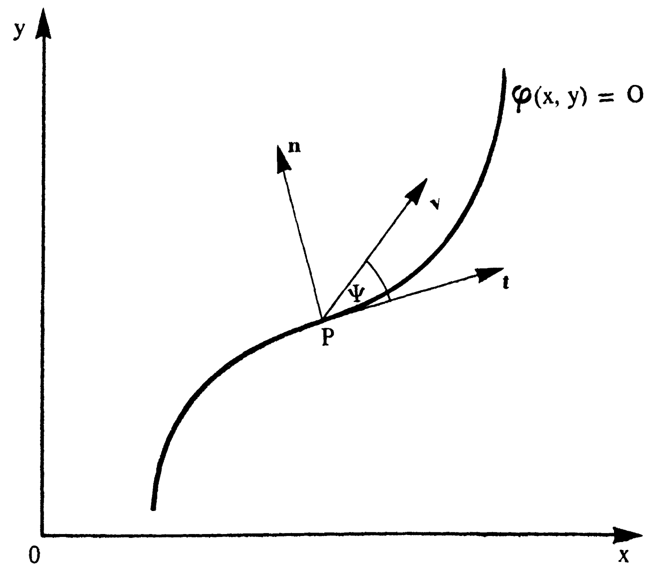

FIG. 22

Choosing a point P of the curve $\phi = 0$, differencing this result across the spatial discontinuity line there and denoting the jump in derivatives with respect to ϕ by δ, we find that

$$\rho\left(\frac{\partial\phi}{\partial x}\,\delta v_x + \frac{\partial\phi}{\partial y}\,\delta v_y\right) + \left(v_x\frac{\partial\phi}{\partial x} + v_y\frac{\partial\phi}{\partial y}\right)\delta\rho = 0, \quad (45.7)$$

since $\boldsymbol{v}$ and ρ are continuous across $\phi = 0$. Now, since $\phi(x, y) = 0$, it follows that along $\phi = 0$ we must have

$$\frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy = 0. \quad (45.8)$$

However, as $\dfrac{dy}{dx}$ is the gradient of $\phi = 0$, it follows directly from equation (45.8) that $\dfrac{\partial\phi}{\partial x}$ and $\dfrac{\partial\phi}{\partial y}$ are proportional to the direction cosines n_x and n_y, with respect to the x and y-axes, of the normal $\boldsymbol{n}$ to $\phi = 0$ at point P. So, since the equation (45.7) is homogeneous, we may replace $\dfrac{\partial\phi}{\partial x}$ by n_x and $\dfrac{\partial\phi}{\partial y}$ by n_y, when the equation immediately simplifies to the result

$$\rho\delta v_n + v_n\delta\rho = 0, \quad (45.9)$$

where $v_n = \boldsymbol{n}.\boldsymbol{v}$. Introducing the unit vector $\boldsymbol{t}$ that is tangent to the discontinuity line at P, so that $\boldsymbol{n}.\boldsymbol{t} = 0$, we may write $\boldsymbol{v} = v_t\boldsymbol{t} + v_n\boldsymbol{n}$ where $\boldsymbol{v}_t = v_t\boldsymbol{t}$ is the tangential component of velocity along the spatial discontinuity curve (characteristic) $\phi = 0$.

Expressing the two scalar equations corresponding to equation (45.2) in terms of ϕ and y' we find that

$$\left(v_x\frac{\partial\phi}{\partial x} + v_y\frac{\partial\phi}{\partial y}\right)\frac{\partial v_x}{\partial\phi} + \frac{a^2}{\rho}\frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial\phi} + v_y\frac{\partial v_x}{\partial y'} = 0, \quad (45.10)$$

and

$$\left(v_x \frac{\partial\phi}{\partial x} + v_y \frac{\partial\phi}{\partial y}\right)\frac{\partial v_y}{\partial\phi} + \frac{a^2}{\rho}\frac{\partial\phi}{\partial y}\frac{\partial\rho}{\partial\phi} + v_y \frac{\partial v_y}{\partial y'} + \frac{a^2}{\rho}\frac{\partial\rho}{\partial y'} = 0. \tag{45.11}$$

Differencing these equations across $\phi = 0$ at point P and replacing $\frac{\partial\phi}{\partial x}$ by n_x and $\frac{\partial\phi}{\partial y}$ by n_y as before then yields

$$v_n \delta v_x + \frac{a^2}{\rho} n_x \delta\rho = 0, \tag{45.12}$$

$$v_n \delta v_y + \frac{a^2}{\rho} n_y \delta\rho = 0. \tag{45.13}$$

The coefficients of equations (45.9), (45.12) and (45.13) must then satisfy the following characteristic determinant if the equations are to have a non-trivial solution:

$$\begin{vmatrix} v_n & \rho n_x & \rho n_y \\ \frac{a^2}{\rho} n_x & v_n & 0 \\ \frac{a^2}{\rho} n_y & 0 & v_n \end{vmatrix} = 0. \tag{45.14}$$

Since $\boldsymbol{n}$ is a unit vector this leads to the characteristic relation

$$v_n(v_n^2 - a^2) = 0. \tag{45.15}$$

Hence, assuming that $\phi = 0$ is not a contact discontinuity for which $v_n = 0$, equation (45.15) shows that

$$v_n^2 = a^2. \tag{45.16}$$

Letting the angle between $\boldsymbol{v}$ and $\boldsymbol{t}$ be ψ (see Fig. 22), we find that

$$v_n = v \sin\psi, \tag{45.17}$$

and so equation (45.16) becomes

$$\left(\frac{v}{a}\right)^2 = \frac{1}{\sin^2 \psi}. \tag{45.18}$$

However, the left-hand side of this equation is simply the square of the **Mach number** M of the flow at point P and so we finally obtain

$$M^2 = \frac{1}{\sin^2 \psi}. \tag{45.19}$$

It is an immediate consequence of this equation that for ψ to be a real angle, and so for real spatial discontinuities (characteristics) to exist, we must have $M \geqq 1$. This proves that real characteristics exist, and so equations (45.1,2) form a hyperbolic system, only in the case of **supersonic flow.** When $M<1$ the flow is **subsonic,** the characteristics become imaginary and the equations then form an **elliptic** system.

Equation (45.18) has a simple geometrical interpretation in supersonic flow which we now explain since it will be of value later in connection with magnetohydrodynamic steady flow. For a fixed point source P of weak disturbances (sound waves with sound speed a) located in a uniform medium at rest, the wavefront at any time t will be a sphere of radius at. This obvious result also follows from equation (25.31b) as the limit of c_f, as $\boldsymbol{H}$ tends to zero. The result in two-dimensional flow will thus be a circle centred at P with radius at. Hence, if the point source P moves with speed v, during time t the point P will move a distance vt. It is thus clear that, relative to P, the disturbances propagating from the source will lie along the tangent lines to a circle of radius at, that pass through the point P distant vt from the centre of the circle (Fig. 23).

The wedge semi-angle $\widehat{OPA}$ (cone semi-angle in three-dimensions) in two-dimensional flow is just the **Mach angle**

ψ. The lines *PA* and *PB* are lines of weak discontinuity and are called **Mach lines.** When the flow is three-dimensional the disturbance lies on the **Mach cone** formed by rotating Fig. 23 about the axis of symmetry *OP*. The region within

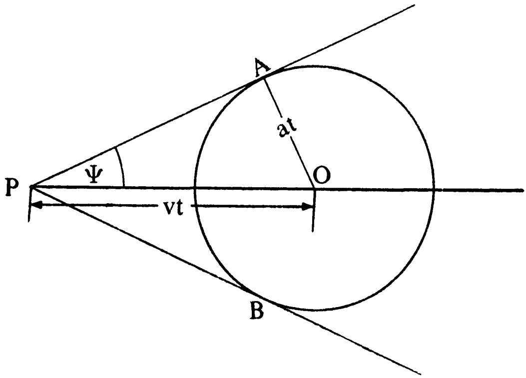

FIG. 23

the Mach cone is the disturbed region while the region surrounding it is at rest. The case of subsonic flow corresponds to the point *P* lying inside the circle of radius *at*. It is then obvious that no real tangents to the circle (characteristics or Mach lines) can exist for subsonic flow.

§ 46. **Magnetohydrodynamic steady parallel flow.** Let us now generalise the ideas of the previous section to the case of the spatial discontinuities that can occur in steady compressible flows of a perfectly conducting fluid in which the magnetic vector $\boldsymbol{H}$ is always parallel to the velocity vector $\boldsymbol{v}$. As in our discussion of incompressible steady flows in § 17, we shall refer to these as **parallel flows.**† Assuming a polytropic gas, then for isentropic flow, the

† Some writers call these **aligned field** flows.

steady state Lundquist equations become

$$\operatorname{div}(\rho \boldsymbol{v}) = 0, \tag{46.1}$$

$$(\boldsymbol{v}\,.\,\operatorname{grad})\boldsymbol{v} + \frac{a^2}{\rho}\operatorname{grad}\rho + \frac{\mu}{4\pi\rho}\boldsymbol{H}\times\operatorname{curl}\boldsymbol{H} = \boldsymbol{0}, \tag{46.2}$$

$$\operatorname{curl}(\boldsymbol{v}\times\boldsymbol{H}) = \boldsymbol{0}, \tag{46.3}$$

$$\operatorname{div}\boldsymbol{H} = 0. \tag{46.4}$$

It is easy to see that if λ is some constant of proportionality, the expression

$$\boldsymbol{H} = \lambda\rho\boldsymbol{v} \tag{46.5}$$

is a solution of equations (46.1), (46.3) and (46.4). Consequently, we may replace equations (46.1) to (46.4) by the equations

$$\operatorname{div}(\rho \boldsymbol{v}) = 0, \tag{46.1}$$

$$(\boldsymbol{v}\,.\,\operatorname{grad})\boldsymbol{v} + \frac{a^2}{\rho}\operatorname{grad}\rho + \frac{\lambda^2\mu}{4\pi}\boldsymbol{v}\times(\operatorname{curl}(\rho\boldsymbol{v})) = \boldsymbol{0}. \tag{46.6}$$

The characteristic equations for these steady state equations may be deduced in a similar fashion to those of the previous section. Indeed, the characteristic equation corresponding to equation (46.1) is identical with the one corresponding to equation (45.1) which was found to be

$$v_n\delta\rho + \rho\delta v_n = 0. \tag{46.7}$$

The characteristic equations corresponding to equation (46.6) can easily be shown to be (see Example 1, § 50)

$$\rho v_n \delta v_n + \rho\boldsymbol{v}_t\,.\,\delta\boldsymbol{v}_t + a^2\delta\rho = 0 \tag{46.8}$$

and

$$\left(1 - \frac{\lambda^2\mu}{4\pi}\rho\right)\delta\boldsymbol{v}_t - \frac{\lambda^2\mu}{4\pi}\boldsymbol{v}_t\delta\rho = \boldsymbol{0}. \tag{46.9}$$

The characteristic relation that is derived from the vanishing of the determinant of the coefficients of equations

(46.7), (46.8) and (46.9) is then

$$v_n^2 - \left(1 - \frac{\lambda^2\mu}{4\pi}\rho\right)a^2 - \frac{\lambda^2\mu}{4\pi}\rho v^2 = 0. \tag{46.10}$$

In terms of the Alfvén number $A = v/b$ and the Mach number $M = v/a$ this equation may be written

$$\frac{M^2A^2}{(M^2+A^2-1)} = \frac{1}{\sin^2\psi} \tag{46.11}$$

where, as before, ψ is the angle between $\boldsymbol{v}$ and $\boldsymbol{t}$ so that

$$v_n = v \sin\psi. \tag{46.12}$$

Equation (46.11) is the magnetohydrodynamic steady parallel flow equivalent of equation (45.19) to which it tends as $\boldsymbol{H} \to \boldsymbol{0}$, for then the Alfvén number $A \to \infty$.

In order that the parallel flow should be hyperbolic, and so have real characteristics, it is necessary that

$$0 \leqq \sin^2\psi \leqq 1$$

for then ψ will be real. Consequently, for real characteristics, we must have

$$0 \leqq (M^2+A^2-1)/M^2A^2 \leqq 1. \tag{46.13}$$

So we have either,

case (f):

$$A > 1, \quad M > 1, \tag{46.14a}$$

or

case (s):

$$A < 1, \quad M < 1 \text{ with } A^2+M^2-1 > 0. \tag{46.14b}$$

The inequality (46.14a) implies that case (f) corresponds to the fast wave since it implies the condition $v > \max(a, b)$. Similarly inequality (46.14b) implies that case (s) corresponds to the slow wave since it implies the condition $v < \min(a, b)$. The regions of validity of these inequalities is indicated by the shaded regions in Fig. 24.

When M or A are such that the flow occurs in an unshaded region no real characteristics exist and the system of equations (46.1) and (46.6) becomes elliptic. In the next section we shall examine the geometrical interpretation of these results.

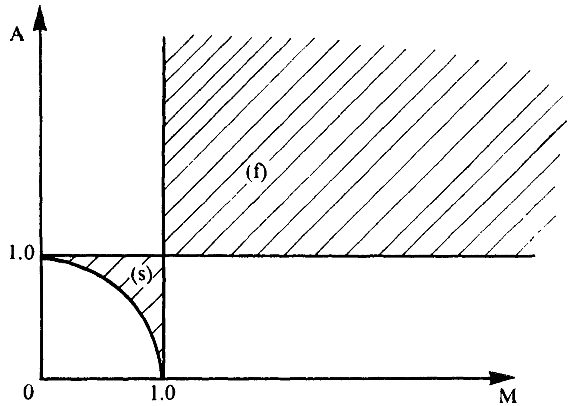

FIG. 24

§ 47. Geometrical construction of spatial discontinuities in parallel flow. The spatial discontinuities that occur in parallel flow can be constructed geometrically in a manner exactly similar to that used in ordinary supersonic flow and illustrated in Fig. 23. However in this case, instead of a circular wavefront diagram, we must use the appropriate wavefront diagram from Fig. 15. To be specific we shall consider Fig. 15(c) for which $s > 1 (s = a^2/b^2)$, the case $s < 1$ being similar.

For waves expanding from a point source into a constant state the disturbance fronts will preserve their initial shape, expanding an amount proportional to the elapsed time t. This situation is illustrated in Fig. 25 which shows the shapes of the fast and slow wavefronts at time $t = 1$.

Since the flow is parallel flow, the characteristics corresponding to a point source moving with velocity $\boldsymbol{v}_0$ along the x-axis will be the tangents to these wavefront curves drawn from a point P on the x-axis distant from O an amount $v_0 t$ in the sense of $-\boldsymbol{v}_0$ (cf., the argument in connection with

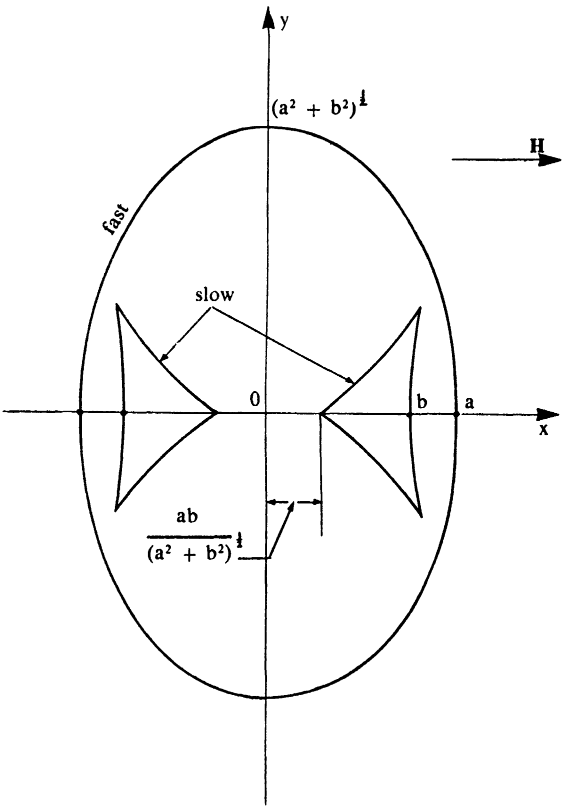

Fig. 25

Fig. 23). Inspection of Fig. 25 shows that we must distinguish four distinct cases, according as:

(a) $$0 < v_0 < \frac{ab}{(a^2+b^2)^{\frac{1}{2}}},$$

(b) $$\frac{ab}{(a^2+b^2)^{\frac{1}{2}}} < v_0 < b,$$

(c) $$b < v_0 < a,$$

(d) $$a < v_0.$$

Case (*b*) corresponds to the end of the velocity vector falling within the slow wavefront and case (*d*) corresponds to it falling outside the fast wavefront.

Now, introducing the Mach number $M = v_0/a$ and the Alfvén number $A = v_0/b$, inequality (*b*) becomes

$$\left(1+\frac{M^2}{A^2}\right)^{-\frac{1}{2}} < A < 1,$$

or

$$A < 1, \quad A^2+M^2-1 > 0.$$

However, since $a > b$, it follows that $M < A$ and so in case (*b*) we find that

$$M < 1, \quad A < 1, \quad A^2+M^2-1 > 0. \tag{47.1}$$

Thus our geometrical arguments have resulted in conditions (46.14*b*) for slow waves.

If we now consider case (*d*) we immediately see that $M > 1$ and, since $a > b$, we have

$$M > 1, \quad A > 1. \tag{47.2}$$

This is just condition (46.14*a*) for fast waves.

In the intervals specified by inequalities (*a*) and (*c*) the geometrical construction shows that no real characteristics exist.

This construction, independently due to H. Grad and

W. R. Sears, has been used in Fig. 26 to show the characteristics that occur in parallel flow and how they are attached to a slender insulating body in steady motion. The cases (*a*), (*b*), (*c*) and (*d*) in Fig. 26 correspond to the cases just discussed.

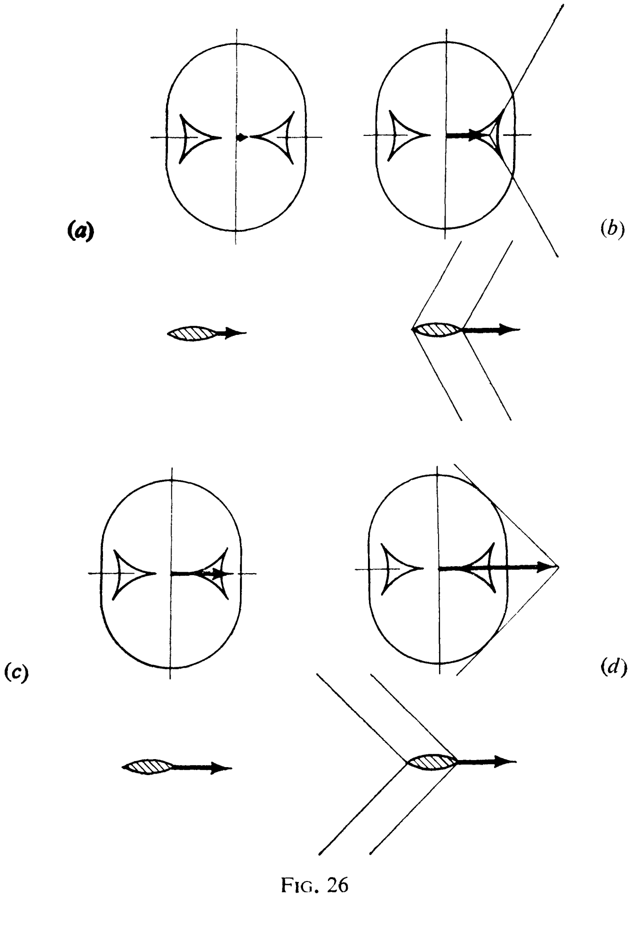

FIG. 26

It should be noticed that in case (*b*), unlike anything that occurs in ordinary gas dynamics, the characteristics are inclined **upstream.**

§ 48. Geometrical construction of spatial discontinuities in arbitrary two-dimensional flow. The construction that was used to study parallel flow in the previous section extends directly to arbitrary two-dimensional steady flow. It must suffice here that we show the possible characteristics and their local behaviour in the vicinity of a slender insulating body in steady motion with velocity $\boldsymbol{v}_0$.

Again we use the wavefront diagram from Fig. 15 that is appropriate to the flow and draw from the origin a line of length v_0 in the direction $-\boldsymbol{v}_0$. The characteristics are again obtained by forming the tangents to the fast and slow wavefronts from this point. The results of this construction are shown in Fig. 27.

A comparison of Figs. 26 and 27 shows the degenerate behaviour of the parallel flow case, but the remarkable feature of upstream characteristics is exhibited by both flows.

It is possible to solve general flow problems of this type by using the notion of generalised Riemann invariants, a form of which was introduced previously in connection with simple waves. Although we shall not examine this problem further it is nevertheless worth drawing attention to the effect of constant components of the magnetic field and velocity normal to the plane of the flow. When no such components are present, the characteristic relation is of fourth degree as indicated in Fig. 27 by the four real characteristics in the purely hyperbolic regions. However, when they are present there is an essential difference and the characteristic relation becomes one of the sixth degree, corresponding to the additional occurrence of transverse disturbances with the consequent complication of the solution. The construction of the characteristics in this case involves the construction of characteristic cones. The local shape of such a cone is obtained by forming the ruled surface comprising all the tangents to the wavefront surface from

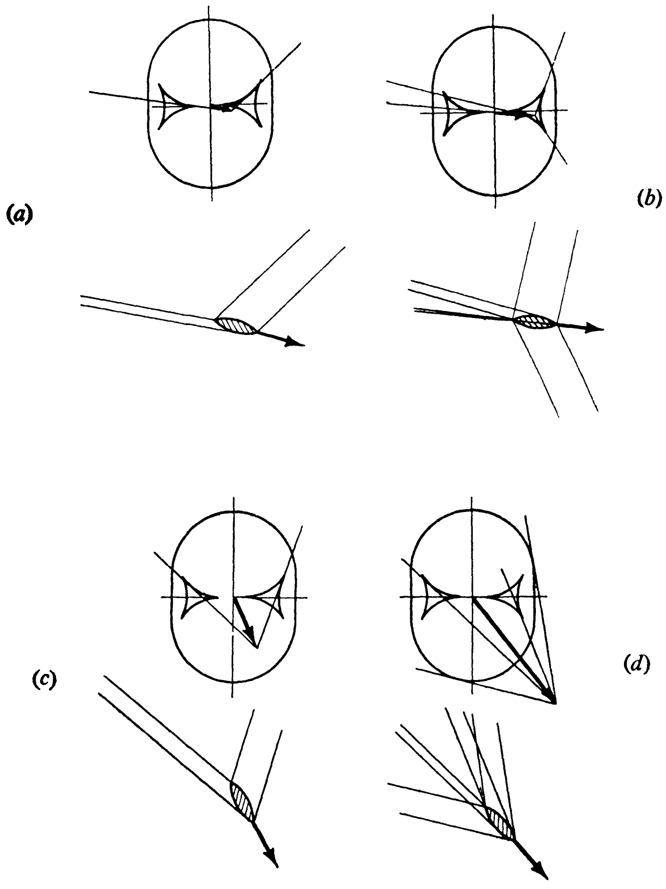

FIG. 27

the end point of the reversed velocity vector $-\boldsymbol{v}_0$. In this case the wavefront surface is, of course, the surface obtained by rotating the appropriate wavefront diagram of Fig. 15 around its x-axis (i.e., revolving it about $\boldsymbol{H}$).

§ **49. Discontinuities in the static case.** When no motion occurs (i.e., $\boldsymbol{v} = \boldsymbol{0}$) the Lundquist equations reduce to

$$\text{grad } p + \frac{\mu}{4\pi} \boldsymbol{H} \times \text{curl } \boldsymbol{H} = \boldsymbol{0} \tag{49.1}$$

and

$$\text{div } \boldsymbol{H} = 0. \tag{49.2}$$

Even these simplified equations allow spatial discontinuities to occur as we shall now show. If, at some point of interest P, we orient the x-axis so that it is directed along the normal $\boldsymbol{n}$ to such a discontinuity the equations become locally one-dimensional in terms of the independent variable x and simplify to (cf., the one-dimensional equations at the start of § 25):

$$\frac{\partial p}{\partial x} \boldsymbol{n} + \frac{\mu}{4\pi} \boldsymbol{H} \times \left(\boldsymbol{n} \times \frac{\partial \boldsymbol{H}}{\partial x} \right) = \boldsymbol{0} \tag{49.3}$$

and

$$\frac{\partial H_x}{\partial x} = \frac{\partial H_n}{\partial x} = 0. \tag{49.4}$$

Differencing these equations across the static discontinuity surface then gives

$$\boldsymbol{n} \delta p + \frac{\mu}{4\pi} \boldsymbol{H} \times (\boldsymbol{n} \times \delta \boldsymbol{H}) = \boldsymbol{0} \tag{49.5}$$

and

$$\delta H_n = 0, \tag{49.6}$$

where, as before, δ signifies the jump in the normal derivative of the associated function.

We may write equation (49.5) in the alternative form

$$\boldsymbol{n}\delta\left(p+\frac{\mu H^2}{8\pi}\right)-H_n\delta \boldsymbol{H}=0. \tag{49.7}$$

Taking the scalar product of this equation with $\boldsymbol{n}$ and using equation (49.6) we then see that

$$\delta\left(p+\frac{\mu H^2}{8\pi}\right)=\delta p^*=0. \tag{49.8}$$

Now, since $\delta H_n = 0$, it follows that

$$\delta \boldsymbol{H} = \boldsymbol{t}\delta H_t+\boldsymbol{n}\delta H_n$$

becomes

$$\delta \boldsymbol{H} = \boldsymbol{t}\delta H_t, \tag{49.9}$$

where δH_t is the tangential component of $\delta \boldsymbol{H}$ with respect to the normal $\boldsymbol{n}$, and $\boldsymbol{n}.\boldsymbol{t} = 0$. However, unless $\delta H_t = 0$, in which case no discontinuity exists, equations (49.7) to (49.9) yield the result that the static discontinuity surface (characteristic surface) at point P is given by

$$H_n = 0. \tag{49.10}$$

Since P was an arbitrary point of the static discontinuity surface it follows immediately that equation (49.10) is valid over the entire static discontinuity surface.

Let us denote the surface by the relation

$$\phi(x, y, z) = \text{constant}. \tag{49.11}$$

The normal $\boldsymbol{n}$ to the surface at any point is then proportional to grad ϕ. Consequently, equation (49.10) may be expressed in the alternative form

$$\boldsymbol{H}.\text{grad}\phi = 0, \tag{49.12}$$

implying that the discontinuity surface is a magnetic surface that is composed of magnetic lines of force.

Taking the scalar product of equation (49.1) with $\boldsymbol{H}$ gives

$$\boldsymbol{H}.\operatorname{grad}p = 0, \tag{49.13}$$

showing that the equi-pressure surfaces $p =$ constant are also magnetic surfaces. However, since by virtue of equation (2.9)

$$\frac{4\pi}{c}\boldsymbol{j} = \operatorname{curl} \boldsymbol{H}, \tag{49.14}$$

we find, directly from equation (49.1), that

$$\boldsymbol{j}.\operatorname{grad}p = 0. \tag{49.15}$$

Equations (49.12), (49.13) and (49.15) imply that magnetic surfaces and equi-pressure surfaces are also current surfaces. So, across such static discontinuity surfaces, a discontinuity in pressure and magnetic field may exist provided that the pressure p^* is continuous across the surface as required by equation (49.8).

This important conclusion forms the basis of experimental attempts at plasma containment purely by means of magnetic fields. A full discussion of this important topic involves consideration of the complicated stability problems of magnetically confined plasmas and will not be attempted here. We can, however, give a brief illustration of a typical and fundamental problem.

Let us consider the special case in which an incompressible plasma of density ρ which is bounded by the plane $x = 0$ is supported against gravity by a vacuum magnetic field which does not penetrate the plasma region. The situation is illustrated in Fig. 28(*a*) in which the shaded region denotes the undisturbed plasma and the dotted region denotes the magnetic field directed normal to the plane of the diagram.

Then, from condition (49.8), on the interface $x = 0$ we

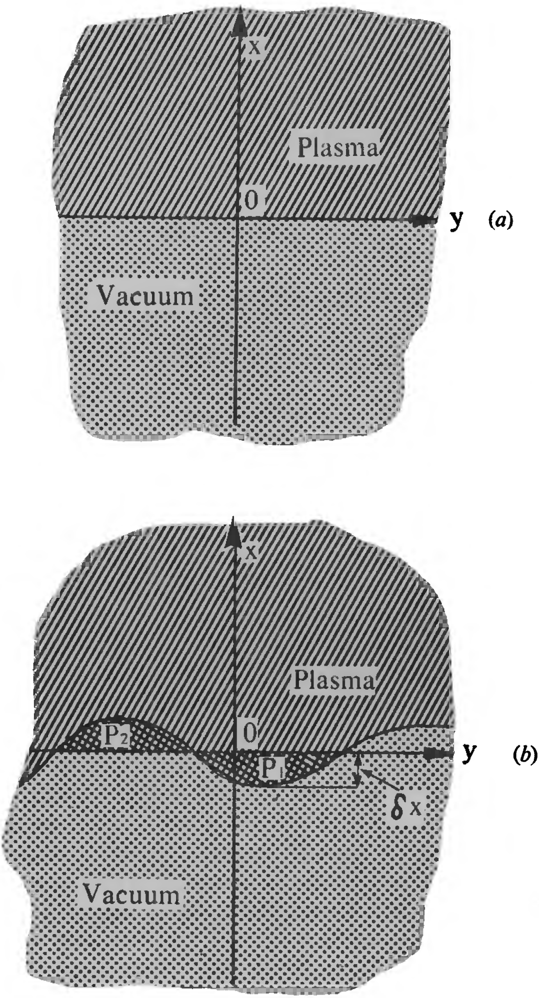

FIG. 28

must have the boundary condition

$$p = \frac{\mu \boldsymbol{H}^2}{8\pi}, \tag{49.16}$$

where p is the plasma pressure immediately above the interface and $\boldsymbol{H}$ is the magnetic field strength in the vacuum region. If, now, the plasma equilibrium is altered so that the plasma moves normal to the magnetic field as in Fig. 28(*b*), so that the shaded regions P_1 and P_2 are of equal area, the plasma will not be compressed and the lines of force will remain unbent. Consequently the magnetic energy will remain constant, while the pressure at the bottom of region P_1 increases by an amount $\rho g \delta x$, whereas the pressure at the top of P_2 decreases by an equal amount, where δx is a small displacement normal to the interface. Since the forces act to increase the size of the ripples on the interface, the configuration must be unstable. This type of instability is usually called the **flute type instability.**

This problem is the magnetohydrodynamic analogue of the well-known **Rayleigh-Taylor** instability experienced by a dense fluid supported against gravity by a less dense fluid. The magnetic field in the vacuum region plays the part of the less dense fluid.

A more realistic configuration is that which occurs in the so-called **pinch effect** in which the magnetic field due to a current I flowing along the surface of a cylindrical plasma column of radius R compresses the plasma. The magnetic field is circumferential and does not exist inside the plasma column; the surface magnetic field H_R at radius R being simply

$$H_R = \frac{2I}{R}. \tag{49.17}$$

Consequently, if the plasma pressure p is constant, equilibrium will result when the magnetic pressure at the exterior surface of the column balances the internal hydrostatic pressure. For equilibrium, the boundary condition (49.16) thus becomes

$$p = \frac{\mu I^2}{2\pi R^2}. \tag{49.18}$$

This simple result illustrates an important form of plasma instability that can occur in plasma columns. Since the magnetic pressure varies inversely as R^2, should any part of the column have a radius slightly less than R, the magnetic pressure at that point will exceed the hydrostatic pressure thereby causing the column to collapse.

An interesting special case of equation (49.1) occurs when the field is **force-free** and the electromagnetic body force $\frac{1}{c}\boldsymbol{j} \times \boldsymbol{H}$ vanishes. This implies that $\boldsymbol{H} \times \text{curl}\,\boldsymbol{H}$ vanishes and equations (49.1,2) simplify to

$$\text{grad}\,p = 0, \quad \text{div}\,\boldsymbol{H} = 0. \tag{49.19}$$

Since when the body force vanishes everywhere the current vector $\boldsymbol{j}$ must always be parallel to the magnetic vector $\boldsymbol{H}$, it follows that for a force-free field we may write

$$\text{curl}\,\boldsymbol{H} = \lambda \boldsymbol{H}, \tag{49.20}$$

where λ is some function of position.

Taking the curl of this equation and using the result $\text{div}\,\boldsymbol{H} = 0$ then yields

$$\nabla^2 \boldsymbol{H} + \lambda^2 \boldsymbol{H} = \boldsymbol{H} \times \text{grad}\,\lambda. \tag{49.21}$$

When λ is constant this becomes the vector Helmholtz equation

$$\nabla^2 \boldsymbol{H} + \lambda^2 \boldsymbol{H} = \boldsymbol{0}. \tag{49.22}$$

It is possible to express the general solution of equation (49.22) in terms of an arbitrary constant unit vector $\boldsymbol{m}$ and a solution u to the corresponding scalar Helmholtz equation (see Example 4, § 50).

§ 50. Examples

1. Derive the characteristic relations for steady parallel magnetohydrodynamic flow:

$$\rho\delta v_n + v_n\delta\rho = 0,$$

$$v_n\delta v_n + \boldsymbol{v}_t \cdot \delta\boldsymbol{v}_t + \frac{a^2}{\rho}\,\delta\rho = 0,$$

$$\left(1 - \frac{\lambda^2\mu}{4\pi}\,\rho\right)\delta\boldsymbol{v}_t - \frac{\lambda^2\mu}{4\pi}\,\boldsymbol{v}_t\delta\rho = \mathbf{0},$$

where $\boldsymbol{H} = \lambda\rho\boldsymbol{v}$ and $\boldsymbol{v} = v_t\boldsymbol{t} + v_n\boldsymbol{n}$. Deduce from the last equation that the flow and the magnetic field do not rotate and show that

$$\frac{\lambda^2\mu}{4\pi}\,\delta(\rho\boldsymbol{v}_t) = \delta\boldsymbol{v}_t,$$

and hence that

$$\delta H_n = 0 \text{ and } \frac{\lambda\mu}{4\pi}\,\delta\boldsymbol{H}_t = \delta\boldsymbol{v}_t.$$

2. Consider steady parallel magnetohydrodynamic flow and by introducing ψ, the angle between the velocity vector $\boldsymbol{v}$ and the tangential vector $\boldsymbol{t}$ to the discontinuity surface, show that the characteristic equations together with the result

$$\sin^2\psi = \frac{M^2 + A^2 - 1}{M^2A^2}$$

yield

$$(v_n^2 - a^2)\,\frac{\delta\rho}{\rho} = \boldsymbol{v}_t \cdot \delta\boldsymbol{v}_t$$

and

$$(v_n^2 - a^2)/v^2 = \sin^2\psi - M^{-2} = (M^2 - 1)/M^2A^2.$$

Hence show that for a compressive change ($\delta\rho > 0$),

$$\boldsymbol{v}_t . \delta\boldsymbol{v}_t > 0 \quad \text{if} \quad M > 1$$

and

$$\boldsymbol{v}_t . \delta\boldsymbol{v}_t < 0 \quad \text{if} \quad M < 1.$$

Selecting the direction of $\boldsymbol{t}$ such that $v_t > 0$ show that

$$\delta v_t > 0, \quad \delta H_t > 0 \quad \text{when} \quad M > 1$$

and

$$\delta v_t < 0, \quad \delta H_t < 0 \quad \text{when} \quad M < 1.$$

Hence since $\delta H_n = 0$, show that for a compressive change the magnetic pressure p_m increases when $M > 1$ and decreases when $M < 1$.

3. Prove that if u is a solution of the scalar Helmholtz equation

$$\nabla^2 u + \lambda^2 u = 0,$$

and $\boldsymbol{m}$ is a constant unit vector, the vectors

$$\boldsymbol{X} = \text{curl}\,(\boldsymbol{m}u), \;\; \boldsymbol{Y} = \frac{1}{\lambda}\,\text{curl}\,\boldsymbol{X}$$

are independent solutions of the vector Helmholtz equation

$$\nabla^2 \boldsymbol{H} + \lambda^2 \boldsymbol{H} = \boldsymbol{0}$$

involving a solenoidal vector $\boldsymbol{H}$.

Hence, by showing that

$$\text{curl}\,(\boldsymbol{X} + \boldsymbol{Y}) = \lambda(\boldsymbol{X} + \boldsymbol{Y}),$$

deduce that $\boldsymbol{H}$ has the general solution

$$\boldsymbol{H} = \text{curl}\,(\boldsymbol{m}u) + \frac{1}{\lambda}\,\text{curl}\,\text{curl}\,(\boldsymbol{m}u).$$

SOLUTIONS TO EXAMPLES

Most of the examples given at the end of each chapter are phrased in such a way that it is not necessary to provide a solution. Answers for the remainder are given below.

CHAPTER III

8. $1/eR_h$.

CHAPTER IV

3. $\lambda = v$, $\lambda = v \pm a$. When $\lambda = v$ the characteristic equations describe a contact discontinuity and $dv = 0$, $a^2 d\rho + \dfrac{\partial p}{\partial S}\, dS = 0$ with dS arbitrary. When $\lambda = v \pm a$ the characteristic equations reduce to $\pm a d\rho + \rho dv = 0$ and $dS = 0$.

5. $\lambda^2 = c^2/\varepsilon\mu$. The characteristic surface is a sphere of radius $ct/\sqrt{\varepsilon\mu}$ at time t.

11. Transverse wave: $x = R_0 \cos\theta \pm bt$, $y = R_0 \sin\theta$. These equations represent two circles each of radius R_0 moving with speeds $\pm b$ along the x-axis.

Fast wave:

$$\frac{x}{b} = \frac{R_0 \cos\theta}{b} \pm \cos\theta \left\{ \frac{c_f}{b} - \frac{s \sin^2\theta}{(c_f/b)[(1+s)^2 - 4s\cos^2\theta]^{\frac{1}{2}}} \right\} t,$$

$$\frac{y}{b} = \frac{R_0 \sin\theta}{b} \pm \sin\theta \left\{ \frac{c_f}{b} + \frac{s \cos^2\theta}{(c_f/b)[(1+s)^2 - 4s\cos^2\theta]^{\frac{1}{2}}} \right\} t.$$

CHAPTER V

2. Extrema lie on $\alpha^2\beta = 1$.

7. The piston velocity V is negative and we distinguish two cases. (a) If $V > (v_x)_{\text{cav}}$, then the piston velocity does not exceed the cavitation speed and the steady state lies in the region $\xi > a_0$ while the simple wave extends up to the piston face and occupies the region $V < \xi < a_0$. (b) If $V < (v_x)_{\text{cav}}$, then the piston speed exceeds the cavitation speed. The steady state still lies in the region $\xi > a_0$ but now the simple wave occupies the region $(v_x)_{\text{cav}} < \xi < a_0$ and a cavitation zone extends from this to the piston face occupying the region $V < \xi < (v_x)_{\text{cav}}$.

8. $(v_x)_{\text{cav}} = -2K_1\sqrt{K_2\rho_0}$.

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About the author

Alan Jeffrey is emeritus professor of engineering mathematics at Newcastle. He has had a distinguished career which included work at University of Delaware, Stanford, Wisconsin and City University Hong Kong. He has published 14 books, some at undergraduate level and others at the research monograph level, some with Springer, and his sales records look very good. He also is well known because he edited some important reference works such as the Handbook of Mathematical Formula.

He has given courses on engineering mathematics at UK and US Universities.

The Newcastle University has announced with deep regret the death of Emeritus Professor Alan Jeffrey on Sunday 6 June 2010.
Emeritus Professor Jeffrey was appointed to the Chair of Engineering Mathematics on 1 September 1963.During his career he held the post of Head of Engineering Mathematics. He retired on 30 September 1984 after which time the title of Emeritus Professor was conferred on him.